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ON TOPOLOGICAL DIVISORS OF ZERO IN TOPOLOGICAL ALGEBRAS

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Abstract. Given a locally m-convex algebra, we examine when a quasi-singular element of it becomes a topological divisor of zero, in a appropriate sense, to which one is forced here, for lack of unity in the given algebra. This is achieved for quasi-singular elements of type (A) and (B) in certain locally m-convex algebras. Furthermore, in the context of suitable locally m-convex algebras, topological zero divisors are roots of Šilov points, while the latter characterize also non-removable ideals. On the other hand, a characterization of Šilov points, accordingly of non-removable ideals, is obtained by means of topological divisors of zero, realized by the same net. Finally, we also characterize advertibly complete locally m-convex algebras through topological divisors of zero.

Key words and phrases: Spectral set relative to H, inverting set relative to H, plane (and pseudoplane) subalgebra, locally m-convex algebra, uniform algebra, boundary set, Šilov boundary, superalgebra, non-removable ideal, topological divisor of zero, element of type (A) and of type (B), associated net, equicontinuous set, hereditarily Weierstrass algebra relative to H, Q-algebra.

1. Introduction

Topological divisors of zero in a unital Banach algebra A (not necessarily commutative) are automatically singular (i.e. non-invertible), while the converse is valid for the singular elements of A contained in the topological boundary $bd(A^{*})$ of the group A^{*} of its invertible elements (see, for instance, C.E. RICKART [8] or [9], E.L. STOUT [10], R. LARSEN [5]). Furthermore, in the commutative case, R. ARENS [1] characterized singular elements $x \in A$, as topological divisors of zero if, and only if, their principal ideals xA are non-removable. A generalization of the latter result is given by W. ŻELAZKO [11], in the context of a Banach function algebra A, getting thus that the only non-removable ideals in A are those contained in kernels of Šilov points; now, the latter ideals are also characterized as those maximal (closed) ideals in A consisting of topological divisors of zero realized by a common net. In the same framework, topological divisors of zero were characterized as roots of Šilov points (see GEL'FAND-RAIKOV-SHILOV [2]). A similar characterization was given by K. NISHIZAWA [7] in normed function algebras and by R. LARSEN [5] in a unital semi-simple Banach algebra A, with $\|x\|^2 \leq k\|x^2\|$, $x \in A$, for some $k > 0$.

Our aim in this paper is to treat the above within the context of topological algebras, in general, not necessarily Banach and/or Banach function algebras, as was the case hitherto. In particular, concerning the aforementioned RICKART's result, by using the notion of a *pseudoplane subalgebra* (cf. (2.5)), we get a generalization of it in the context of a locally m-convex algebra (E, Γ), such that (E, p) is a pseudoplane subalgebra of its completion for some continuous submultiplicative semi-norm p on E. Namely, every quasi-singular element $x \in E$ contained in the topological boundary of its quasi-invertible elements implies that $1 - x$ is a topological divisor of zero, provided its *associated net* is eventually bounded in (E, Γ), being actually valid in (E, p) (cf. (3.9), Theorem 3.1, Proposition 3.1 and Corollary 3.2).

On the other hand, we consider topological divisors of zero as roots of Šilov points in appropriate locally m-convex algebras, while the converse is also examined in another type of locally m-convex algebras (Theorem 4.1); the compilation of both would lead to normed algebras (!), so that we actually consider another version of the previous result (see e.g. Proposition 4.1) that circumvents the normed case. Furthermore, by extending ŻELAZKO's results, we show that the Šilov points, when viewed as closed regular maximal 2-sided ideals, are *non-removable* in a topological algebra E with Šilov boundary $\partial(E)$ contained in the spectrum of any *superalgebra* of E (Lemma 5.1). In general, non-removable ideals are characterized as the ideals contained in kernels of Šilov points, for a suitable topological algebra (Theorem 5.1). Yet, based on Theorems 4.1 and 5.1, the Šilov points are further characterized as those continuous characters, whose kernels consist of topological divisors of zero realized by the same net (Theorem 6.1, Corollary 6.1).

Part of this paper is based on the author's Ph. D. Thesis, written under the guidance of Professor A. MALLIOS (Univ. of Athens), to whom the author wishes to express her sincere thanks for inspiring and helpful discussions, as well as, for constant encouragement during this work.

2. Preliminaries

In all that follows, by a *topological algebra* we mean a complex topological algebra E with non empty *spectrum* $\mathfrak{M}(E)$ endowed with the *Gel'fand topology*. The *Gel'fand map* of E is given by

$$\mathcal{G}: E \to \mathcal{C}(\mathfrak{M}(E)) : x \mapsto \mathcal{G}(x) \equiv \hat{x}$$

(2.1)

$$: \mathfrak{M}(E) \to \mathbf{C} : f \mapsto \hat{x}(f) := f(x).$$

The image of $\mathcal{G}$, denoted by $E^\wedge$, is called the *Gel'fand transform algebra* of E and is topologized as a *locally m-convex algebra* by

$$E^\wedge \subseteq \mathcal{C}_c(\mathfrak{M}(E)), \tag{2.2}$$

where the algebra $\mathcal{C}_c(\mathfrak{M}(E))$ carries the topology "c" of compact convergence (cf. [6: p. 140]). Given a subspace H of E, a subset $B \subseteq \mathfrak{M}(E)$ is said to be a *spectral set of E relative to H*, if

$$Sp_E(x) = \hat{x}(B), \tag{2.3}$$

for every $x \in H$, where $Sp_E(x)$ denotes the *spectrum of* $x \in H$ (cf. [6: p. 47; Definitions 1.1, 1.2]). If E is unital, a $B \subseteq \mathfrak{M}(E)$ with the property

(2.4) $$\forall\ x \in H \ \text{ with } \ 0 \notin \hat{x}(B) \Rightarrow x \in E^{\cdot},$$

where $E^{\cdot}$ stands for the multiplicative group of invertible elements of E, is called an *inverting set of E relative to H.* Evidently, in a unital topological algebra E, every spectral set relative to a subspace H is an inverting set relative to H, while the converse statement holds true if H contains the constants [3: p: 15, Theorem 2.1]. Of course, every inverting set of E (relative to E) is such a one relative to H.

On the other hand, a *subalgebra F* of E is *pseudoplane*, if

(2.5) $$E^{\circ} \cap F \subseteq F^{\circ},$$

that is, every $x \in F$ quasi-invertible in E (viz. $x \in E^{\circ}$) has its quasi-inverse in F. Yet, by a *uniform algebra* we mean a locally m-convex algebra E such that a fundamental family of submultiplicative semi-norms, say $\Gamma = \{p\}$, defining its topology, satisfies the property

(2.6) $$p(x^2) = p(x)^2, \qquad x \in E,$$

for every $p \in \Gamma$ (cf. [6: p. 274, Definition 5.1]).

Now, we say that a closed $B \subseteq \mathfrak{M}(E)$ is a *boundary set of E*, if for every $x \in E$ there exists $f \in B$ such that

(2.7) $$|\hat{x}(f)| = \sup_{h \in \mathfrak{M}(E)} |\hat{x}(h)| \equiv p_{\mathfrak{M}(E)}(\hat{x}).$$

The smallest such set, denoted by $\partial(E)$, is defined as the *Šilov boundary of E* (cf. [6: p. 189, Definitions 2.1, 2.2]) and its existence is accomplished in a unital topological algebra E with compact spectrum $\mathfrak{M}(E)$ (cf. [6: p. 192, Corollary 2.1]). On the other hand, considering for every $x \in E$ the set

(2.8) $$M_{\hat{x}} = \{f \in \mathfrak{M}(E) : |\hat{x}(f)| = \sup_{h \in \mathfrak{M}(E)} |\hat{x}(h)|\},$$

being closed by the continuity of $\hat{x}$, an element $f \in \mathfrak{M}(E)$ *belongs to* $\partial(E)$ *if, and only if, for every open neighbourhood V of f in* $\mathfrak{M}(E)$ *there exists* $x \in E$ *such that*

(2.9) $$M_{\hat{x}} \subseteq V$$

(cf. [6: p. 190, Lemma 2.1]). A *superalgebra F* of E is, of course, a topological algebra F containing a subalgebra isomorphic to E with an eventual stronger topology from the relative one; we write $E \subset F$. *A* 2-sided ideal I in E is said to be *non-removable*, if for every superalgebra F of E, the 2-sided ideal (I) generated by I is proper in F.

However, we also apply in the sequel (cf. Thoerem 5.1) the more general notion of a *triple* (E, ϕ, F), with $\phi : E \to F$ a *continuous algebra morphism* between the *topological algebras* E, F. (*I* owe this to A. MALLIOS). In this context, we also speak of a *non-removable ideal I of E*, whenever $\phi(I)$ generates a proper ideal of F, as before. *Henceforth by an ideal we always mean a 2-sided one.*

3. Topological divisors of zero.

Given a topological algebra E, a *2-sided topological divisor of zero* is an element $x \in E$ such that there exists a net $(z_\delta) \subseteq E$, $z_\delta \not\to 0$ with

$$\lim_\delta (xz_\delta) = 0 = \lim_\delta (z_\delta x). \tag{3.1}$$

The set of 2-sided topological divisors of zero in E is denoted by $\mathcal{B}_0(E)$. *In the sequel, by a topological divisor of zero we mean a 2-sided one.* In this respect, we note that *in a unital topological algebra E, a topological divisor of zero is singular*, i.e. non-invertible. Accordingly, *in a non-unital topological algebra E an element* $x \in E$ *is quasi-singular*, i.e. non-quasi-invertible, *if* $1-x$ *is a topological divisor of zero in the unitization of E*, $E_1 \equiv E \oplus \mathbf{C}$. By setting

$$\mathcal{B}_0^q(E) = \{x \in E : 1 - x \in \mathcal{B}_0(E_1)\}, \tag{3.2}$$

we seek conditions guaranteeing the converse of the last statement, viz. when a quasi-singular element belongs to $\mathcal{B}_0^q(E)$.

In this concern, an *element* $x \in E$ is said to be *of type (A)*, if it is quasi-singular and there exists a net $(x_\delta) \subseteq E^\circ$, with $x = \lim_\delta x_\delta$; that is

$$x \in \overline{E^\circ} \cap (E^\circ)^c. \tag{3.3}$$

The set of the elements of type (A) is denoted by $t_A(E)$. Thus, one has the following fact, whose proof is straightforward.

Lemma 3.1. *In every Q-algebra E one has*

$$t_A(E) = bd(E^\circ). \tag{3.4}$$

(We denote by $bd(E^\circ)$ the *topological boundary of* E°). ∎

On the other hand, one also obtains the next.

Lemma 3.2. *Let E be a topological algebra such that*

(3.5) *there exists a continuous submultiplicative semi-norm p such that (E,p) is a pseudoplane subalgebra of its completion $(\tilde{E}, \tilde{p})$.*
We say, for short, that E is *p-pseudoplane.*

Then, for every $x \in t_A(E,p)$, the respective net $(x_\delta^\circ)_{\delta \in J}$ is frequently unbounded (i.e., not eventually bounded [6: p. 466, Remark]) *in (E,p), hence in E, as well. Therefore, a f o r t i o r i, (x_δ°) is also unbounded in E.*

Proof. Assuming that the net $(x_\delta^\circ)_{\delta \in J}$ is eventually bounded in $(\tilde{E}, \tilde{p})$, we get a contradiction: Indeed, by the assumption, for a given $\varepsilon > 0$, there exist $\lambda > 0$ and $\delta_1 \in J$ such that

$$\tilde{p}(x_\delta^\circ) \leq \lambda \cdot \varepsilon, \tag{3.6}$$

for every $\delta \geq \delta_1$. Hence by (3.6) and since $x_\delta \to x \in (E,p) \underset{\to}{\subseteq} (\tilde{E}, \tilde{p})$, one gets $\tilde{p}(x_\delta^\circ \circ x) \leq \tilde{p}_1(1 - x_\delta^\circ)\tilde{p}(x - x_\delta) \leq (1 + \tilde{p}(x_\delta^\circ))\tilde{p}(x - x_\delta) \leq (1 + \lambda\varepsilon)\tilde{p}(x - x_\delta) < 1$, for every $\delta \geq \delta'$ for suitable $\delta' \in J$. (Here p_1 stands for the semi-norm in the unitization, associated to p). Thus, for the complete locally m-convex algebra $(\tilde{E}, \tilde{p})$ one has $x_\delta^\circ \circ x \in (\tilde{E}, \tilde{p})^\circ$, for some $\delta \geq \delta'$ (cf. [6: p. 101, Corollary 6.3]), such that $x \in (\tilde{E}, \tilde{p})^\circ$. Hence, by (3.5) (cf. also (2.5)), $x \in (E,p)^\circ$, a contradiction to the hypothesis for x.

Therefore, *the net $(x_\delta^\circ)_{\delta \in J}$ is frequently unbounded in $(\tilde{E}, \tilde{p})$, hence in* (E,p), as well. Otherwise, for every neighbourhood $U_p(\varepsilon) \ni 0 \in (E,p)$, hence for every neighbourhood $\tilde{U}_{\tilde{p}}(\varepsilon) = \overline{U_p(\varepsilon)} \ni 0 \in (\tilde{E}, \tilde{p})$ (cf. [4: p. 134]), there would exist $\lambda > 0$ and $\delta_0 \in J$ with $\tilde{p}(x_\delta^\circ) = p(x_\delta^\circ) \leq \lambda \cdot \varepsilon$, for every $\delta \geq \delta_0$, so that (x_δ°) would be eventually bounded in $(\tilde{E}, \tilde{p})$, a contradiction. Furthermore, (x_δ°) is *frequently unbounded in E,* by the continuity of *p.* ■

Remarks 3.1. – i) Since in a topological algebra E one has

$$t_A(E) \subseteq t_A(E,p), \ \forall \ p \text{ continuous submultiplicative semi-norm on } E, \tag{3.7}$$

the conclusion of Lemma 3.2 holds true for every $x \in t_A(E)$.

ii) Yet, we remark (A. MALLIOS) that every *p*-pseudoplane locally *m*-convex algebra is also Γ-pseudoplane. (See next corollary).

iii) The previous proof yields also, implicitly, that; *the limit of quasi-invertible elements whose net of quasi-invertibles is eventually bounded, is quasi-invertible as well.* (See also [9 : p. 18, Theorem 1.4.21]).

By applying a similar proof to that one of Lemma 3.2, one has the following.

Corollary 3.1. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally m-convex algebra which is a pseudoplane subalgebra of its completion.* (We say, for short, that E is Γ-pseudoplane). *Then, for every $x \in t_A(E)$, the respective net $(x_\delta^\circ)_{\delta \in J}$ is frequently unbounded in E.* ∎

Scholium 3.1. – It is obvious that every semi-normed advertibly complete algebra fulfills the condition of Lemma 3.2. On the other hand, we can also give the following example (due initially to A. MALLIOS): Let $(E, \|\cdot\|)$ be a normed Q-algebra. Then, E equipped with the initial locally m-convex topology, defined by the set $\Gamma \equiv \{p\}$ of all submultiplicative semi-norms on E (being the stronger one on E, cf. [6: p. 11]), becomes a locally m-convex algebra with $p \equiv \|\cdot\| \in \Gamma$, such that (E, p) is a pseudoplane subalgebra of its completion $(\tilde{E}, \tilde{p})$ (cf. [6 : p. 96, Corollary 5.1, along with the Remark before it]).

According to Lemma 3.2, by taking $x \in t_A(E, p)$, the respective net $(x_\delta^\circ)_{\delta \in J}$ is frequently unbounded in (E, p), hence, there exists $\varepsilon > 0$, such that for every $\lambda > 0$ one has

$$p(x_{\delta'}^\circ) > \varepsilon \cdot \lambda, \tag{3.8}$$

with $\delta' \in J'$ cofinal subset of J. Thus, *every* $x \in t_A(E, p)$, hence also every $x \in t_A(E)$ (cf. (3.7)), *is associated with a net* of the form

$$z_{\delta'} := (p(x_{\delta'}^\circ))^{-1} \cdot x_{\delta'}^\circ, \ \delta' \in J' \subseteq J, \text{ with } J' \text{ cofinal}, \tag{3.9}$$

called *associated net of x*.

The above lead us to the following: If $x \in t_A(E)$, for E a topological algebra, then, by definition, there exists a net $(x_\delta^\circ) \subseteq E$, with (x_δ) converging to x. Now, assuming that this net is frequently unbounded, by analogy with (3.9) and applying, in place of p therein, the Minkowski functional of a neighbourhood of zero, we find what we may call here too an *associated net of* x.

Thus, we now set.

Definition 3.1. Let E be a topological algebra. We say that an *element* $x \in t_A(E)$ is *of type (B)*, whenever any associated net of x is eventually bounded. We denote the set of these elements of E by $t_B(E)$.

In this regard, we remark that the previous definition has a more concrete form in the case E is a Γ-pseudoplane locally m-convex algebra (cf. Corol. 3.1).

Now, on the basis of the previous discussion, we have a condition quaranteeing when a particular type of quasi-singular elements are *topological divisors of zero, in the sense that they belong to* $\mathcal{B}_0^q(E)$ (cf. (3.2)). In this concern, we refer to [8: p. 1066, Theorem 2.6, (iii)], or [9: p. 22, Theorem 1.5.4, (iii), p. 24, Theorem 1.5.9, (ii)] and [7: p. 4, Lemma 1] for the case of unital commutative Banach algebras.

Now, the next result specializes to Theorem 2.7 of E. L. STOUT [10: p. 10].

Theorem 3.1. *Let E be a Γ-pseudoplane locally m-convex algebra. (Take, equivalentrly [6], E advertibly complete). Then, every $x \in E$ of type (B) belongs to $\mathcal{B}_0^q(E)$.*

Proof. Assuming that an element $x \in E$ is of type (B), then, by definition, for every $\alpha \in I$, there exist $\lambda_\alpha > 0$ and $\delta_0' \in J'(\subseteq J,\text{cofinal})$, with (cf. (3.9) by also taking $p \equiv p_0$)

$$p_\alpha(z_{\delta'}) \leq \lambda_\alpha, \quad p_\alpha \in \Gamma, \tag{3.10}$$

for every $\delta' \geq \delta_0'$. Since $p_0(z_{\delta'}) = 1, \quad \delta' \in J'$, *the associated net* $(z_{\delta'})_{\delta' \in J'}$ *of* x *diverges from zero*. Furthermore, since by hypothesis x is of type (A) and due to the relations (3.8), (3.9), (3.10), one gets for every $\alpha \in I$,

$$\begin{aligned} p_\alpha((1-x)z_{\delta'}) &= \lim_\delta p_\alpha((1-x_\delta)z_{\delta'}) = \lim_\delta [p_\alpha(x_{\delta'}^\circ - x_\delta x_{\delta'}^\circ)p_0(x_{\delta'}^\circ)^{-1}] \\ &\leq p_\alpha(x_{\delta'}^\circ - x_{\delta'} x_{\delta'}^\circ)p_0(x_{\delta'}^\circ)^{-1} = p_\alpha(x_{\delta'})p_0(x_{\delta'}^\circ)^{-1} \xrightarrow[\delta']{} 0, \end{aligned} \tag{3.11}$$

yielding that $1-x$ is a topological divisor of zero. ■

As we shall see in the following, there is no distinction between elements of type (A) and (B) in (E,p) as in (3.5). That is, one has.

Proposition 3.1. *Let E be a p-pseudoplane (topological) algebra* (cf. (3.5)). *Then, in (E,p) the elements of type (A) and of type (B) coincide, belonging to $\mathcal{B}_0^q(E,p)$, as well. That is, we have*

$$t_A(E,p) = t_B(E,p) \subseteq \mathcal{B}_0^q(E,p). \tag{3.12}$$

Proof. Assuming that x is of type (A) in (E,p), the net $(x_\delta^\circ)_{\delta \in J}$ is frequently unbounded in (E,p) (cf. Lemma 3.2). Thus, by considering the associated net

$(z_{\delta'})_{\delta' \in J'}$ of x, we have $p(z_{\delta'}) = 1$, $\delta' \in J'$, so that $(z_{\delta'})$ *is bounded in* (E, p); hence, eventually bounded, so x is of type (B) in (E, p). Now, by applying (3.11) to the particular p involved herein, we get our claim for $\mathcal{D}_0^q(E, p)$. ■

As a consequence of the previous Proposition 3.1 and Remark 3.1, i), we obtain the following.

Corollary 3.2. *Let E be a p-pseudoplane topological algebra. Then, for every $x \in E$ of type (A), $1 - x$ is a topological divisor of zero in $(E, p)_1$. That is,*

$$t_A(E) \subseteq \mathcal{D}_0^q(E, p). \quad ■ \tag{3.13}$$

In fact, the relation (3.12) characterizes the algebra considered as a p-pseudoplane algebra, according to the next result, whose a consequence is, for that matter, Corollary 3.2.

Theorem 3.2. *Let E be a topological algebra and p a continuous submultiplicative semi-norm on E. Then, the following assertions are equivalent:*

1) E is p-advertibly complete.

2) E is p-pseudoplane.

3) $t_A(E, p) \subseteq \mathcal{D}_0^q(E, p)$.

Proof. $1) \Leftrightarrow 2)$: Apply [6: p. 96, Corollary 5.1 and previous Remark] for the semi-normed algebra (E, p). $2) \Rightarrow 3)$, by Proposition 3.1. Conversely, if E is not p-pseudoplane, there exists $x \in E$ such that $x \in (\tilde{E}, \tilde{p})^{\circ} \cap ((E, p)^{\circ})^c \subseteq \overline{(E, p)^{\circ}} \cap ((E, p)^{\circ})^c$; hence, $x \in t_A(E, p)$ and by hypothesis $x \in \mathcal{D}_0^q(E, p) \subseteq \mathcal{D}_0^q(\tilde{E}, \tilde{p})$, a contradiction. ■

On the other hand, one still obtains.

Theorem 3.3. *In a locally m-convex algebra $(E, \Gamma = \{p\})$ the following statements are equivalent:*

1) E is advertibly complete.

2) E is Γ-pseudoplane.

3) $t_A(E) \subseteq \bigcup_{p \in \Gamma} \mathcal{D}_0^q(E, p)$.

Proof. $1) \Leftrightarrow 2)$: See [6: p. 96, Corollary 5.1 and Remark preceding it]. Assuming 2), for $x \in t_A(E)$, the respective net $(x_\delta^{\circ})_{\delta \in J}$ is frequently unbounded (Corollary 3.1). So there exists $p \in \Gamma$ for which the respective associated net is bounded in (E,p) (see also (3.9)), hence, by definition, x is of type (B) in (E,p). Therefore (Theorem 3.1, we actually apply an analogous proof of it), $x \in \mathcal{B}_0^q(E,p)$, so that 2) $\Rightarrow$ 3). Conversely, suppose that 3) holds true and E is not Γ-pseudoplane. Then, by applying a similar proof to that one of Theorem 3.2, there exists $x \in (\tilde{E})^{\cdot} \cap (E^{\cdot})^c \subseteq t_A(E)$, so that, by hypothesis, $x \in \mathcal{B}_0^q(E,p)$, for some $p \in \Gamma$, hence, in $\mathcal{B}_0^q(\tilde{E},\tilde{p})$ too, that is a contradiction, so that 3) $\Rightarrow$ 2), as well. ■

Motivation for the above Theorem 3.3 has been a relevant result due to A. BEDDAA, *Charactérisations des Q-algèbres localement multiplicativement convexes* (manuscript).

4. Characterization of topological divisors of zero

In this section we give a characterization of topological divisors of zero in suitable topological algebras by means of Šilov points. An analogous result has been given by I. GEL'FAND - D. RAIKOV - G. SHILOV [2], C.E. RICKART [9] and R. LARSEN [5] in Banach function algebras and by K. NISHIZAWA [7] in normed function algebras.

Theorem 4.1. *Let E be a topological algebra with Šilov boundary $\partial(E)$ and $x \in E$. Yet, consider the following two assertions:*

1) $x \in \mathcal{B}_0(E)$.
2) *There exists $f \in \partial(E)$ such that $f(x) = 0$.*

Then, if E has the initial topology of the map $r \circ \mathcal{G} : E \to \mathcal{C}_c(\partial(E))$ (see (4.1) below), *1)⇒2). On the other hand, if $\partial(E)$ is compact and E carries the initial topology of its Gel'fand map $\mathcal{G}$, then 2)⇒1).*

Proof. Let $x \in \mathcal{B}_0(E)$ with $0 \notin \hat{x}(\partial(E))$. Considering the algebra $\mathcal{C}_c(\partial(E))$, by hypothesis one has $\hat{x}|_{\partial(E)} \in \mathcal{C}(\partial(E))^{\cdot}$ and there exists $(z_\delta)_{\delta\in J} \subseteq E$, $z_\delta \not\to 0$ with $\lim_\delta xz_\delta = 0 (= \lim_\delta z_\delta x)$. The continuity of the (composite) map

$$(4.1) \qquad E \xrightarrow{\mathcal{G}} E^\wedge \xrightarrow{r} \mathcal{C}_c(\partial(E)) : x \mapsto \hat{x} \mapsto \hat{x}|_{\partial(E)},$$

implies $0 = r(\mathcal{G}(\lim_\delta xz_\delta)) = \lim_\delta \widehat{xz_\delta}|_{\partial(E)} = \lim_\delta (\hat{x}|_{\partial(E)} \cdot \hat{z}_\delta|_{\partial(E)})$, with $\hat{z}_\delta|_{\partial(E)} \not\to 0$, so that $\hat{x}|_{\partial(E)} \in \mathcal{B}_0(\mathcal{C}_c(\partial(E)))$, a contradiction.

On the other hand, by taking $0 \neq x \in E$ satisfying 2), consider for every $n \in \mathbf{N}$ the set

$$(4.2) \qquad U(f,x,n) = \left\{ g \in \mathfrak{M}(E) : |\hat{x}(g)| < \frac{1}{n} \right\},$$

open neighbourhood of the given $f \in \partial(E)$. Then, the set Δ of all pairs $\delta = (U, n)$ as in (4.2), is directed with partial order $\delta_1 \leq \delta_2$ if $U_2 \subseteq U_1$ and $n_1 \leq n_2$. Hence (cf. (2.9)), for every $\delta = (U, n)$, there exists $x_\delta \in E$ such that

$$M_{\delta_\delta} \subseteq U(f, x, n), \tag{4.3}$$

providing a net $(x_\delta)_{\delta \in \Delta}$ in E. By assuming

$$\|\hat{x}_\delta\|_\infty \equiv p_{\mathfrak{M}(E)}(\hat{x}_\delta) = \sup_{h \in \mathfrak{M}(E)} |\hat{x}_\delta(h)| = p_{\partial(E)}(\hat{x}_\delta) = 1, \tag{4.4}$$

and by taking a suitable power of x_δ, we can have

$$|\hat{x}_\delta(h)| < \frac{1}{\lambda \cdot n}, \tag{4.5}$$

for every $h \notin U(f, x, n)$, with $\lambda = \|\hat{x}\|_\infty \equiv p_{\mathfrak{M}(E)}(\hat{x}) = \sup_{h \in \mathfrak{M}(E)} |\hat{x}(h)|$. (For simplicity we do not write the power of x_δ. We further remark that (4.4) is preserved under an eventual change of x_δ by a suitable power of it, so that (4.5) holds true).

Now, given $\varepsilon > 0$, let $\frac{1}{n_\varepsilon} < \varepsilon$. Then, for every $n \geq n_\varepsilon$, one gets (cf. (4.4) and (4.5))

$$|\hat{x}(h)|\,|\hat{x}_\delta(h)| < \frac{1}{n} \|\hat{x}_\delta\|_\infty < \varepsilon, \quad h \in U(f, x, n), \tag{4.6}$$

while

$$|\hat{x}(h)|\,|\hat{x}_\delta(h)| \leq \lambda \cdot \frac{1}{\lambda \cdot n} < \varepsilon, \quad h \notin U(f, x, n), \tag{4.7}$$

yielding that

$$p_{\mathfrak{M}(E)}(\widehat{xx}_\delta) \equiv \|\widehat{xx}_\delta\|_\infty < \varepsilon, \tag{4.8}$$

for every $n \geq n_\varepsilon$, as above. Hence, there exists $(x_\delta)_{\delta \in \Delta} \subseteq E$ with $\hat{x}_\delta \nrightarrow 0$ and $\widehat{xx}_\delta \to 0$ (cf. (4.4) and (4.8)), proving that x is a topological divisor of zero of E with respect to its initial topology defined by $\mathcal{G}$. ■

In this concern, we further remark that one is finally led to consider a *normed algebra*, in order to have the previous theorem, as a *characterization of* $\mathcal{B}_0(E)$, *in terms of roots of Shilov points.* So (A. MALLIOS), it remains to have the same characterization for *non-normed* algebras. A response to the latter gives the following result in the setting of unital algebras.

Proposition 4.1. *Let E be a unital topological algebra with the initial topology of its Gel'fand map, Šilov boundary $\partial(E)$ and $x \in E$. Moreover, consider the following two assertions:*

1) $x \in \mathcal{B}_0(E)$.

2) $x \in Ker(f)$ for some $f \in \partial(E)$.

Then, if $E^\wedge|_{\partial(E)}$ is plane in $\mathcal{C}_c(\partial(E))$, $1) \Rightarrow 2)$. On the other hand, if $\partial(E)$ is compact, $2) \Rightarrow 1)$.

Proof. The topology considered in E implies that $x \in \mathcal{B}_0(E)$ iff $\hat{x} \in \mathcal{B}_0(E^\wedge)$, where $E^\wedge = E^\wedge|_{\partial(E)}$ up to a continuous algebra isomorphism. Thus, due to the hypothesis for $E^\wedge|_{\partial(E)}$, and arguing as in the proof of Theorem 4.1, $1) \Rightarrow 2)$, we obtain that $\hat{x} \in (E^\wedge)^{\bullet}$, a contradiction. For $2) \Rightarrow 1)$ see Theorem 4.1. ∎

Referring to the assumption of being $E^\wedge|_{\partial(E)} = E^\wedge$ plane in $\mathcal{C}_c(\partial(E))$, we note that this is achieved when $\partial(E)$ is an inverting set, while the same is characterized by the latter property of $\partial(E)$, in the case E is semi-simple [3: p. 29 Theorem 3.1].

Now using the concept of an inverting set we can remove in Proposition 4.1, $1) \Rightarrow 2)$ the compact-open topology from E, taking thus the next alternative form.

Proposition 4.2. *Let E be a unital topological algebra with Šilov boundary $\partial(E)$ and $x \in E$. Moreover, consider the following two assertions:*

1) $x \in \mathcal{B}_0(E)$.

2) $x \in Ker(f)$, for some $f \in \partial(E)$.

Then, if $\partial(E)$ is an inverting set, $1) \Rightarrow 2)$, while $2) \Rightarrow 1)$ if $\partial(E)$ is compact and E has the initial topology of its Gel'fand map. ∎

4(a). **An application.** –As already remarked, in a unital topological algebra E, every inverting set of E is also inverting relative to a subspace H of it. Now, by considering instead a subalgebra F of E, we examine the converse of the latter statement, obtaining thus an extension of a relevant result in [7: p. 3, Theorem 1] for unital commutative Banach algebras, as well as, an application of Corollary 3.2 and Theorem 4.1.

Theorem 4.2. *Let E be a unital p-plane topological algebra, with (E, p) having the initial topology from $\mathcal{C}_c(\partial(E,p))$* (see also (4.1)). *Moreover, let F be a dense subalgebra of E and B a closed equicontinuous subset of $\mathfrak{M}(E)$, such that*

$$\partial(E,p) \subseteq B. \tag{4.9}$$

Then, the following two assertions are equivalent:

1) *B is an inverting set of E.*
2) *B is an inverting set of E relative to F.*

Proof. Obviously, 1)⇒2). On the other hand, assuming 2), let $x \in E$ with $0 \notin \hat{x}(B)$. Since $\overline{F} = E$, there exists $(x_\delta) \subseteq F$ such that $x = \lim_\delta x_\delta$, thus $f(x) = \lim_\delta f(x_\delta)$ with respect to the simple convergence on $B \subseteq \mathfrak{M}(E)$, and hence, by hypothesis for B, with respect to the uniform one. Therefore, since $f(x) \neq 0$, $f(x_\delta) \neq 0$ for every $f \in B$ and $\delta \geq \delta_0$. Then, by 2), $x_\delta \in E^{\cdot}$ for every $\delta \geq \delta_0$; that is (x_δ) is eventually in $E^{\cdot}$. If $x \notin E^{\cdot}$, then, by the preceding, x is an element of type (A) in E, hence (see e.g. Corollary 3.2) a topological divisor of zero in (E, p). So (Theorem 4.1) $\hat{x}$ vanishes at a Šilov point of (E, p), accordingly by (4.9) at a point of B, a contradiction. Thus, $x \in E^{\cdot}$, proving that 2) ⇒1), as well. ■

Referring to (4.9), we note that in a unital topological algebra E, hereditarily Weierstrass relative to a subspace H of E, every closed spectral (equivalently inverting, when $1_E \in H$) set of E relative to H is a boundary set of E relative to H (that is (2.7) is valid only for $x \in H$), hence it contains the Šilov boundary of E relative to H, $\partial_H(E)$ ([3: p. 19, Theorem 2.2]; cf. also [7: p. 2, Remark 3] for function algebras). In this regard, we say that a topological algebra E is *hereditarily Weierstrass relative to a subspace H* of it, if $\mathfrak{M}(E)$ is a Weierstrass space with respect to $|H^\wedge|$, meaning that every $|\hat{x}|$, $x \in H$, attains its supremum at a point of $\mathfrak{M}(E)$, while this property is retained by every closed subset of $\mathfrak{M}(E)$. Furthermore, $\partial_H(E)$ is defined as the least boundary set of E relative to H, in the above sense.

5. Non-removable ideals

In this section we characterize the non-removable 2-sided ideals of an appropriate topological algebra E in terms of its Šilov points, consequently, in view of Corollary 6.1 below, of its topological divisors of zero.

We first have the following.

Lemma 5.1. *Let E be a topological algebra with Šilov boundary $\partial(E)$ such that $\partial(E) \subseteq {}^t\phi(\mathfrak{M}(F))$ for any "superalgebra" F of E (in the general sense of a triple (E, ϕ, F);* cf. Preliminaries*). Then, for every $f \in \partial(E)$,* $\ker(f)$ *is a non-removable ideal of E. Hence, any ideal I of E with $I \subseteq \ker(f)$, $f \in \partial(E)$, is non-removable.*

Proof. For every $F \supset E$ as in the statement, the 2-sided ideal $(\phi(\ker(f)))$ generated by $\phi(\ker(f)), f \in \partial(E)$, is proper in F. Otherwise, there exist e.g. z_i, y_i in a certain F, $x_i \in \ker(f)$, $1 \le i \le n$, such that $\sum_{i=1}^{n} z_i \phi(x_i) y_i = \omega \in F$, hence $g(\omega) = \sum_{i=1}^{n} g(z_i) f(x_i) g(y_i) = 0$, where $f = {}^t\phi(g)$, $g \in \mathcal{M}(F)$, a contradiction. ■

Remark 5.1. –In the previous Lemma 5.1, the condition $\partial(E) \subseteq {}^t\phi(\mathcal{M}(F))$ is satisfied in the case of [6: p. 195, Theorem 2.2]. (But, see also loc. cit., p. 196, Corollary 2.2, in the special case e.g. that E, F are unital commutative locally m-convex Q-algebras).

According to Lemma 5.1, one concludes that every ideal contained in the kernel of a Šilov point is non-removable. In fact, these ideals are the only non-removable ones in the class of topological algebras considered in the following.

Theorem 5.1. *Let E be a topological algebra with Šilov boundary $\partial(E)$. Moreover, assume the following two assertions:*

1) *I is a non-removable ideal of E.*
2) *$I \subseteq \ker(f)$, for some $f \in \partial(E)$.*

Then, if E is semi-simple with continuous Gel'fand map and $\partial(E)$ compact, 1)⟹2). On the other hand, if $\partial(E) \subseteq {}^t\phi(\mathcal{M}(F))$, for any "superalgebra" F of E (see Preliminaries), *then 2)⟹1).*

Proof. 1)⟹2): By hypothesis $\mathcal{C}(\partial(E))$ is a "superalgebra" of E, so if I is a non-removable ideal in E, then $(I) \subsetneq \mathcal{C}(\partial(E))$. Hence, $I \subseteq (I)$ is contained in a maximal closed (since $\mathcal{C}(\partial(E))$ is Banach) ideal of $\mathcal{C}(\partial(E))$, say $M = \ker(f)$, $f \in \partial(E) \cong \mathcal{M}(\mathcal{C}(\partial(E)))$ (cf. [6: p. 221, Corollary 1.2]), so that 1)⟹2).

Now, 2)⟹1) in view of Lemma 5.1. ■

In the previous theorem, 1)⟹2), one can avoid the hypothesis that E is semi-simple at the expense, however, of taking $\phi(I) \subseteq \ker(f)$, where $E \xrightarrow{\phi} \mathcal{C}(\partial(E))$, with $\phi = r \circ \mathcal{G}$ (see (4.1)).

For convenience, we state the same Theorem 5.1 in the following less general form. So we have.

Corollary 5.1. *Let E be a semi-simple topological algebra with Gel' fand map continuous and $\partial(E)$ compact, with $\partial(E) \subseteq {}^t\phi(\mathcal{M}(F))$, for any "superalgebra" F of E (cf. Lemma 5.1). Then, I is a non-removable ideal of E if, and only if, $I \subseteq \ker(f)$, for some $f \in \partial(E)$.* ■

Remark 5.2. –In view of the above Remark 5.1, we still note that the hypothesis of the previous Corollary 5.1 is fulfilled e.g. for any *unital commutative semi-simple locally m-convex Q-algebra E, having* $\partial(E)$ *compact* (with respect to a superalgebra inducing the given topology on E).

6. Characterization of the Šilov boundary

On the basis of Theorems 4.1 and 5.1, we now characterize the Šilov points of a suitable topological algebra E as those continuous characters of E whose kernels consist of topological divisors of zero with respect to the same net in E. This characterization was proved by W. ŻELAZKO [11] in the context of Banach function algebras.

Now, concerning the definition of a non-removable ideal, by restricting ourselves to *commutative unital superalgebras, with* ϕ *a topological algebra isomorphism (into)* (for which Theorem 5.1 is certainly true, as well), we get at the following result. (In fact, this restriction is applied to get 2)$\Rightarrow$1) of the next theorem).

Theorem 6.1. *Let E be a topological algebra with Šilov boundary* $\partial(E)$ *and* $f \in \mathfrak{M}(E)$. *Moreover, assume the following two assertions:*

1) $f \in \partial(E)$.
2) *The elements of* $\ker(f)$ *are topological divisors of zero with respect to the same net* $(x_\delta)_{\delta\in\Delta}$ *in E.*

Then, if $\partial(E)$ *is compact, and E carries the initial topology defined by the Gel'fand map* $\mathcal{G}$, *1)*$\Rightarrow$*2).*

On the other hand, if E is a semi-simple topological algebra with continuous Gel'fand map and $\partial(E)$ *compact, then* 2)$\Rightarrow$1).

Proof. 1)$\Rightarrow$2): It results from 2)$\Rightarrow$1) in Theorem 4.1, by applying an analogous argument, where now the decreasing sequence of the open neighbourhoods considered is not necessarily defined by some root of the given $f \in \partial(E)$.

2)$\Rightarrow$1): By hypothesis and Theorem 5.1, it suffices to prove that $\ker(f)$ is a non-removable ideal in E. (In fact, by Theorem 5.1, one gets $\ker(f) \subseteq \ker(g)$, with $g \in \partial(E)$, so that $f = g$; cf. [6: p. 69, Lemma 7.2]): Assuming that $\ker(f)$ is removable, there exist $x_i \in \ker(f)$, $i = 1,2,\dots,n$ and z_i, $i = 1,\dots,n$, in a certain superalgebra F of E, as above, such that

$$\sum_{i=1}^{n} z_i x_i = 1_F. \tag{6.1}$$

Multiplying (6.1) by $(x_\delta)_{\delta\in\Delta}$, we have

$$x_\delta = \sum_{i=1}^{n} z_i x_i x_\delta \to 0 \tag{6.2}$$

while, by definition, $x_\delta \not\to 0$. ■

Corollary 6.1. *Let E be a topological algebra having the initial topology of its Gel'fand map (equivalently (A. MALLIOS) a uniform with continuous Gel'fand map [6]), $\partial(E)$ compact and $f \in \mathfrak{M}(E)$. Then, the following two assertions are equivalent:*

1) $f \in \partial(E)$.

2) *The elements of* $\ker(f)$, *for* $f \in \mathfrak{M}(E)$, *are topological divisors of zero with respect to the same net* $(x_\delta)_{\delta\in\Delta}$ *in E.* ■

An immediate consequence of the previous Corollary 6.1 in conjunction with Theorem 5.1 (see also the proof 2) $\Rightarrow$ 1) of Theorem 6.1) is the next.

Corollary 6.2. *Let E be a uniform algebra with continuous Gel'fand map and* $\partial(E)$ *compact such that* $\partial(E) \subseteq {}^t\phi(\mathfrak{M}(F))$ (see Theorem 5.1). *Then, the three following assertions are equivalent:*

1) *The ideal* $I \subsetneq E$ *is non-removable.*

2) $I \subseteq \ker(f)$, *for some* $f \in \partial(E)$.

3) *The ideal* $I \subsetneq E$ *consists of topological divisors of zero with common net* $(x_\delta) \subseteq E$. ■

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