Author: Alexander Abian

Title: A Fundamental fixed point Theorem Revisited

Creator: HDML

BULLETIN OF THE GREEK MATHEMATICAL SOCIETY
Vol. 21(1980)

A FUNDAMENTAL FIXED POINT THEOREM REVISITED

By
Alexander Abian

Abstract.

Without using the Axiom of Choice, the existence of a fixed point of a mapping f (having the properties mentioned in the Theorem) is proved by assuming to the contrary and by arriving at the contradiction of constructing a set K of pairwise distinct values of iterations of f such that K implies the existence of the set of all ordinals. Some additional fixed point theorems and their consequences are also derived.

Let f be a mapping from a set E into itself. An element e of E is called a fixed point of f if and only if

$$e = f(e) \tag{1}$$

A major significance of fixed point theorems in mathematics consists in the fact that a solution of an equation

---

This work is partly supported by the Iowa State University SHRI.

of a varied kind (e.g., algebraic, differential, partial differential, set-theoretical, etc.) can be determined as a fixed point of a suitable mapping. Indeed, let

$$Q(x) = 0 \tag{2}$$

be an equation in a context where 0 and additive cancellation are meaningful. Adding x to both sides of (2), we obtain $x=x+Q(x)$. Motivated by the latter, we consider the mapping f defined by

$$f(x) = x+Q(x) \tag{3}$$

Obviously, e is a solution of equation (2) if and only if e is a fixed point of mapping f given by (3).

Thus, any theorem which ensures the existence of a fixed point of a mapping can be used as a method for solving an appropriate equation. Of course, the converse also holds since the determination of a fixed point of a mapping f can be reduced to solving the appropriate equation $x = f(x)$.

Let, as before, f be a mapping from a set E into itself. In our attempt to determine whether or not f has a fixed point, motivated by (1) and as mentioned above, we consider the question of solving the equation

$$x = f(x) \tag{4}$$

Accordingly, we can replace every x in (4) by f(x) and obtain the equation

$$f(x) = f(f(x)) = f^2(x) \tag{5}$$

Again, in view of (4), replacing every x in (5) by f(x), we obtain

$$f(f(x)) = f(f(f(x))) = f^3(x) \quad \text{or} \quad f^2(x) = f^3(x)$$

Continuing in this manner, for every natural number n we obtain

$$f^n(x) = f^{n+1}(x) \quad \text{for every} \quad n < \omega \tag{6}$$

where we let $f^0(x) = x$.

Let us observe that if a is a solution of (6) then $f^n(a)$ is a fixed point of f in which case

$$f^n(a) = f^{n+1}(a) = f^{n+2}(a) = \ldots \tag{7}$$

Motivated by the above, and starting with an arbitrary element a of E, we consider the set I of the values of all the finite interations of f, i.e.,

$$I = \{a, f(a), f^2(a), f^3(a), \ldots, f^n(a), \ldots\}$$

Now, let us suppose additionally that f is such that the assumption that f has no fixed point, in accordance with (7), implies that the elements of I are pairwise distinct and that $E-I \neq \emptyset$. But then letting $b \in (E-I)$ and denoting b by $f^\omega(a)$ we may continue the process of iteration. Again, let us suppose that f is such that the assumption that f has no fixed point implies that $f^r(a)$ is defined for every ordinal r and that $f^r(a) \neq f^s(a)$ for every pair of distinct ordinals r and s of ZF set-theory (which is taken throughout this article as the underlying set-theory). Now, if x is an element of E such that $x = f^r(a)$ for some ordinal r then we associate with x the ordinal r, otherwise, we associate with x the ordinal 0. Thus the assumption that f has no fixed point (without additional supposition on f) implies that with every element of the set E a unique ordinal is associated and that eve-

ry ordinal is associated with some element of E. However, this (since E is a set) implies (by virtue of the Axiom Scheme of Replacement of ZF) that the abovementioned associates of the elements of E form a set, say, S. Clearly, S is the set of all ordinals. But this leads to a contradiction since in ZF there is no set of all ordinals. Indeed, if S existed then (US)+1 would be an ordinal greater than every ordinal in S which is impossible since S supposed to be the set of all ordinals. Hence, our assumption is false and f (under our additional supposition) has a fixed point.

The above outline of the procedure of proof is used in proving the fundamental fixed point Theorem 1 below. First, however, we introduce the following.

Definition 1. A partially ordered set is called A-inductive if and only if every nonempty well ordered subset of it has a least upper bound.

Based on Definition 1 and without the use of the Axiom of Choice, we prove the following.

Theorem 1. *Let $(P,\leq)$ be a nonempty A-inductive partially ordered set. Let f be a mapping from P into P such that*

$$x \leq f(x) \quad \text{for every} \quad x \in P \tag{8}$$

Then f has a fixed point.

Proof. Let us assume tc the contrary that f has no fixed point. Thus,

$$x < f(x) \quad \text{for every} \quad x \in P \tag{9}$$

Let a be an element of P. For every ordinal r we define:

$$f^r(a) = a \quad \text{if} \quad r = 0 \tag{10}$$

$$f^r(a) = f(f^{r-1})(a)) \quad \text{if } r \text{ is a nonzero and nonlimit ordinal} \tag{11}$$

$$f^r(a) = \operatorname{lub}_{i<r} f^i(a) \quad \text{if } r \text{ is a limit ordinal} \tag{12}$$

Since $(P,<)$ is A-inductive, from (10), (11), (12) it follows that for every ordinal r it is the case that $f^r(a)$ is an element of P. Moreover, from (9) we see that for every ordinal r and s

$$r \neq s \quad \text{implies} \quad f^r(a) \neq f^s(a) \tag{13}$$

Now, to every element x of P we assign an ordinal $g(x)$ as follows:

$$g(x) = r \quad \text{if} \quad x = f^r(a), \quad \text{otherwise} \quad g(x) = 0 \tag{14}$$

From (13) it follows that g defined by (14) is an ordinal-valued function whose domain is P. However, since P is a set, from the Axiom Scheme of Replacement of ZF, it follows that the range of g is also a set. But (10), (11), (12) and (14) imply that the range of g must be the set of all ordinals which is a contradiction. Thus, our assumption is false and f has a fixed point.

Based on the above fundamental fixed point theorem and again without the use of the Axiom of Choice, we prove

Theorem 2. *Let $(Q,\leq)$ be a nonempty A-inductive partially ordered set. Let f be an order preserving mapping from Q into Q, i.e.,*

$$x \leq y \quad \textit{implies} \quad f(x) \leq f(y) \quad \textit{for every} \quad x,y \in Q \tag{15}$$

Moreover, let

$$a \leq f(a) \quad \textit{for some} \quad a \in Q \tag{16}$$

Then f has a fixed point.

Proof. Let us consider the subset P of Q given by

$$P = \{x \mid x \in Q \quad \text{and} \quad x \leq f(x)\} \tag{17}$$

From (16) we see that P is a nonempty partially ordered set. From (15) we see that $x \leq f(x)$ implies $f(x) \leq f(f(x))$ and therefore from (17) it follows that f maps P into P. Next, let W be a nonempty well ordered subset of P. Since $(Q, \leq)$ is A-inductive, $w = \text{lub } W$ is an element of Q. Also, since $x \leq w$ for every $x \in W$, from (15) it follows that $f(x) \leq f(w)$ for every $x \in W$. However, since W is a subset of P, from (17) it follows that $x \leq f(x) \leq f(w)$ for every $x \in W$. But then since $w = \text{lub } W$, we see that $w \leq f(w)$, which by (17) implies that $w \in P$. Consequently, P is an A-inductive subset of Q.

From the above it follows that $(P, \leq)$ is a nonempty A-inductive partially ordered set and f maps P into P and $x \leq f(x)$ for every $x \in P$. Thus, by Theorem 1 we see that f has a fixed point, as desired.

The following well known fixed point theorem of Tarski is readily derived from Theorem 2.

Corollary 1. *An order preserving mapping from a nonempty complete partial-ordered set into itself has a fixed point.*

Proof. Clearly, every nonempty complete partially ordered set Q is a priori a nonempty A-inductive partially ordered set. Let a be the minimum of Q. Obviously, $a \leq f(a)$ for every order preserving mapping f from Q into Q. But then the conclusion of the Corollary follows from Theorem 2.

Let us recall that the Axiom of Choice states that if E is a set then there exists a mapping h from E such that

$$h(X) \in X \quad \text{for every} \quad X \in E \quad \text{with} \quad X \neq \emptyset \tag{18}$$

We recall also that a version of Zorn's lemma states that:

Every nonempty A-inductive partially ordered sets has a maximal element. (19)

It is easy to prove that Zorn's Lemma implies the Axiom of Choice (i.e., that (19) implies (18)). Indeed, let H be the set of all mappings h_i from subsets E_i of E satisfying (18) with h replaces by h_i and E replaced by E_i. Clearly, $(H,\subseteq)$ is a nonempty A-inductive partially ordered set and by (19) has a maximal element h. Obviously, h satisfies (18).

On the other hand, the proof of the fact that the Axiom of Choice implies Zorn's lemma can be given based on Theorem 1 as shown below.

Corollary 2. *The Axiom of Choice is equivalent to Zorn's Lemma.*

Proof. We showed above that (19) implies (18). In order to prove that (18) implies (19), let us assume to the contrary. Thus, let $(P,\leq)$ be a nonempty A-inductive partially ordered set and let us assume that $(P,\leq)$ has no maximal element. Consequently, for every element x of P there exists an element y of P such that y succeeds x, i.e., $x<y$. Hence for every element x of P the set X of all the successors of x is nonempty. Let E be the set of all such X's. By (18), there exists a mapping h from E such that $h(X) \in X$ for every $X \in E$. Hence, for every $x \in P$ we see that $x \leq h(X)$. Let f be a mapping from P into P such that $f(x) = h(X)$ for every $x \in P$ and therefore

$x < f(x)$ for every $x \in P$. Since $(P,\leq)$ is nonempty and A-inductive, from Theorem 1 it follows that f has a fixed point. But this leads to a contradiction since $x < f(x)$ for every $x \in P$. Thus, our assumption is false and hence (18) implies (19), as desired.

Remark 1. *Let us observe that the fundamental fixed point Theorem 1 is proved without the use of the Axiom of Choice and it is not at all apparent how the use of the Axiom of Choice could facilitate the proof of Theorem 1. On the other hand, Zorn's lemma (given in version (19)) provides an immediate proof of Theorem 1. Indeed, in Theorem 1, the maximal element of P is a desired fixed point. But curiously enough, the proof that the Axiom of Choice implies Zorn's lemma (in version (19)) is based on the fundamental fixed point Theorem 1.*

Definition 2. A partially ordered set is called A-quasi-inductive if and only if every well ordered subset of it has an upper bound.

Let us recall again that another version of Zorn's lemma states that:

> Every nonempty A-quasi-inductive partially ordered set has a maximal element. (20)

Obviously, (20) implies (19). To show that (19) implies (20), it is enough to consider the set L of all the well ordered subsets of a partially ordered set and partial order L by the initial segment relation and apply the hypothesis of (20) to the conclusion of (19).

In contrast to the case of Theorem 1, we prove Theorem 2 below inevitably by using Zorn's lemma (conveniently in

version (20)).

Theorem 2. *Let f be a mapping from a nonempty A-quasi-inductive partially ordered set $(P^*,\leq)$ into itself such that $x \leq f(x)$ for every $x \in P^*$. Then f has a fixed point.*

Proof. By (20) we see that $(P^*,\leq)$ has a maximal element which obviously is a fixed point of f.

Remark 2. *The above proof of Theorem 2 suggests that Theorem 2 can be proved by using the Axiom of Choice (as stated in (18)). However, again, this is accomplished via Theorem 1. This reflecte again the fact that the proof that the Axiom of Choice implies Zorn's lemma (in version (20)) is based on the fundamental fixed point Theorem 1.*

No References are given since the article is self-con - tained.

Department of Mathematics
Iowa State University
Ames, Iowa 50011
U.S.A.