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S-WEAKLY HAUSDORFF SPACES

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Abstract.

In this paper semi convergence and semi open sets are used to define and investigate the s-weakly Hausdorff separation axiom. Characterizations of s-weakly Hausdorff spaces are given and the relationships between this new separation axiom and other separation axioms are investigated.

1. Introduction.

Semi open sets were first introduced and investigated in 1963 [5]. In 1975 semi-Ti, i=0,1,2, and s-regular spaces were defined in [8] and [6], respectively, by replacing the word open in the definitions of Ti, i=0,1,2, and regular by semi open, respectively, and it was shown that semi-Ti is strictly weaker than Ti, i=0,1,2, semi-Ti implies semi-Ti - 1,i=1,2, and s-regular is strictly weaker than regular. Listed below are other definitions and theorems that will be utilized in this paper.

Definition 1.1. If (X,T) is a space and $A\subset X$, then A is semi open, denoted by $A\in SO(X,T)$, iff there exists $O\in T$ such $O\subset A\subset \bar{O}$ [5].

Definition 1.2. Let (X,T) be a space and let $A,B\subset X$. Then A is semi closed iff $X-A$ is semi open and the semi closure of B, denoted by scl B, is the intersection of all semi closed sets containing B [1].

Definition 1.3. Let (X,T) be a space and let R be the equivalence relation on X defined by xRy iff $\overline{\{x\}} = \overline{\{y\}}$. Then the T_0-identification space of (X,T) is $(X_0 . S_0)$, where X_0 is the set of equivalence classes of R and S_0 is the decomposition topology on X_0, which is T_0 [11].

Definition 1.4. A net $\{x_\alpha\}_{\alpha\in A}$ semi converges to x in a space (X,T), denoted by $\{x_\alpha\}_{\alpha\in A}\xrightarrow{s}x$, iff $\{x_\alpha\}_{\alpha\in A}$ is eventually in every semi open set containing x. If $\{x_\alpha\}_{\alpha\in A}$ is a net in (X,T), let $S\lim\{x_\alpha\}_{\alpha\in A} = \{x\in X \mid \{x_\alpha\}_{\alpha\in A}\xrightarrow{s} x\}$ [2].

Definition 1.5. A space (X,T) is weakly Hausdorff iff $\overline{\{x\}} = \overline{\{y\}}$, whenever there exists a net converging to both x and y [4].

Definition 1.6. A space (X,T) is s-weakly Hausdorff iff $\overline{\{x\}} = \overline{\{y\}}$, whenever there exists a net semi converging to both x and y.

Definition 1.7. A space (X,T) is semi-R_1 iff for $x,y\in X$ such that scl $\{x\}\neq$ scl $\{y\}$ there exist disjoint semi open sets U and V such that scl $\{x\}\subset U$ and scl $\{y\}\subset V$ [2].

Definition 1.8. A space (X,T) is semi-R_0 iff for each $O \in SO(X,T)$ and each $x \in O$, $\text{scl}\ \{x\} \subset O$ [7].

Definition 1.9. A subset A of a space (X,T) is an α-set iff $A \subset \text{Int}(\text{Int}(A))$ [9].

Definition 1.10. A space (X,T) is extremely disconnected iff for each $O \in T$, $\bar{O} \in T$ [11].

Theorem 1.1. *If A is an α-set in a space (X,T) and $U \in SO(X,T)$, then $U \cap A \in SO(X,T)$* [9].

Theorem 1.2. *If (X,T) is a space, $A\ Y\ X$, and $A \in SO(X,T)$, then $A \in SO(Y,T_Y)$* [5].

Theorem 1.3. *The natural map $P:(X,T) \to (X_0,S_0)$ is continuous, closed open, and $P^{-1}(P(O)) = O$ for all $O \in T$* [3].

Theorem 1.4. *For each $a \in A$ let (X_a,T_a) be a space and let $\phi \neq A_a \subset X_a$ such that $A_a = X_a$ except for finitely many $a \in A$. Then $\prod_{a\in A} A_a \in SO(\prod_{a\in A} X_a, P)$, where P denotes the product topology on $\prod_{a\in A} X_a$, iff $A_a \in SO(X_a,T_a)$ for all $a \in A$* [10].

2. Characterizations.

Theorem 2.1. *The following are equivalent: (a) (X,T) is s-weakly Hausdorff, (b) if $\{x_a\}_{a\in A}$ is a semi convergent net in X, then $S\ \lim\ \{x_a\}_{a\in A} \subset C_x$ for some $x \in X$, where C_x is the equivalence class of R containing x, (c) if $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, then there exist disjoint semi open sets U and V such that $x \in U$ and $y \in V$, (d) if $x,y \in X$ such that $\overline{\{x\}} \neq$ $\neq \overline{\{y\}}$, then there exist disjoint open sets U and V such that $\overline{\{x\}} \subset \bar{U}$ and $\overline{\{y\}} \subset \bar{V}$, (e) for each $x \in X$, $\overline{\{x\}} = \bigcap_{A \in O_x} \text{scl}\ A$, where*

$O_x = \{A \subset X \mid \overline{\{x\}} \subset A \in SO(X,T)\}$, (f) for each α-set Y, (Y,T_Y) is s-weakly Hausdorff, and (g) (X_0,S_0) is s-weakly Hausdorff.

Proof. It follows immediately from the definition that (a) implies (b).

(b) implies (c): Assume there exist $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$ and if $U,V \in SO(X,T)$ such that $x \in U$ and $y \in V$, then $U \cap V \neq \phi$. Then $x \neq y$. Let $D = \{O \in T \mid x,y \in \bar{O}\}$ and let $\geq$ be the binary operation on D defined by $O_1 \geq O_2$ iff $O_1 \subset O_2$. Let O_1, $O_2 \in D$. Then $x,y \in O_1 \cap O_2 = C$, for suppose not. If $x \in C$, then $y \in X-C$, where C and X-C are disjoint semi open sets, which is a contradiction. Thus $x \notin C$ and by a similar argument $y \notin C$ but then $x \in (O_1-C) \cup \{x\} \in SO(X,T)$ and $y \in (O_2-C) \cup \{y\} \in SO(X,T)$, which is a contradiction. Thus $x,y \in O_1 \cap O_2$, which implies $O_1 \cap O_2 \in D$. Hence $(D,\geq)$ is a directed set. For each $O \in D$ let $x_O \in O$. Let $U \in SO(X,T)$ such that $x \in U$. Let $V \in T$ such that $V \subset U \subset \bar{V}$. Then $y \in \bar{V}$ and $V \in D$ such that if $W \in D$ and $W \geq V$, then $x_W \in U$, which implies $\{x_O\}_{O \in D} \overset{s}{\to} x$. Similarly $\{x_O\}_{O \in D} \overset{s}{\to} y$ and $x,y \in C_z$ for some $z \in X$, but then $\overline{\{x\}} = \overline{\{y\}}$, which is a contradiction.

(c) implies (d): The straightforward proof is omitted.

(d) implies (e): Let $x \in X$. Let $y \notin \overline{\{x\}}$. Then $\overline{\{x\}} \neq \overline{\{y\}}$ and there exist disjoint open sets U and V such that $x \in \bar{U}$ and $y \in \bar{V}$. Then $V \cup \{y\}$, $U \cup \overline{\{x\}} \in SO(X,T)$ and $\mathrm{scl}(U \cup \overline{\{x\}}) \subset X - (V \cup \{y\})$, which implies $y \notin \bigcap_{A \in O_x} \mathrm{scl}\, A$.

(e) implies (f): Let Y be an α-set and let $\{x_\alpha\}_{\alpha \in A}$ be a net in Y semi converging to both $x,y \in Y$ using the relative topology on Y. Then $\overline{\{x\}}_Y = \overline{\{y\}}_Y$, for suppose not, say $x \notin \overline{\{y\}}_Y$. Then $x \notin \overline{\{y\}}_X$ and there exists $O \in O_y$ such that $x \notin \mathrm{scl}_X O$. By Theorem 1.1 $U = Y \cap (X - \mathrm{scl}_X O)$, $V = Y \cap O \in SO(X,T)$ and by Theorem 1.2. $U,V \in SO(Y,T_Y)$, but then U and V are disjoint semi open sets in Y containing x and y, respectively, which is a contradiction. Hence, (Y,T_Y) is s-weakly Hausdorff.

(f) implies (g): Since X is an α-set, then (X,T) is s-weakly Hausdorff. For each $z \in X$ let C_z be the equivalence class of R containing z. Let $C_x, C_y \in X_0$ such that $\overline{\{C_x\}} \neq \overline{\{C_y\}}$ Then $C_x \neq C_y$, which implies $\overline{\{x\}} \neq \overline{\{y\}}$. Thus there exist disjoint open sets U and V in X such that $x \in \bar{U}$ and $y \in \bar{V}$. Since $P:(X,T) \to (X_0,S_0)$ is continuous, closed, open, and $P^{-1}(P(O)) = O$ for all $O \in T$, then P(U) and P(V) are disjoint open sets in X_0, $C_x \in P(\bar{U}) = \overline{P(U)}$, and $C_y \in P(\bar{V}) = \overline{P(V)}$. Hence, by the arguments above, (X_0,S_0) is s-weakly Hausdorff.

(g) implies (a): For each $z \in X$ let C_z be the equivalence class of R containing z. Let $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Then $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$, say $x \in \overline{\{y\}}$. Since f is closed, then $\overline{\{C_y\}} \subset P(\overline{\{y\}})$ and since $C_x \cap \overline{\{y\}} = \phi$, then $C_x \notin \overline{\{C_y\}}$ and $\overline{\{C_x\}} \neq \overline{\{C_y\}}$. .Then there exist disjoint semi open sets U and V in X_0 such that $C_x \in U$ and $C_y \in V$. Since f is continuous and open, then $P^{-1}(U)$ and $P^{-1}(V)$ are disjoint semi open sets in X containing x and y, respectively. Thus, by the arguments above, (X,T) is s-weakly Hausdorff.

Theorem 2.2. *If (X,T) is s-weakly Hausdorff and* T_0, *then (X,T) is semi-*T_2.

Proof. If $x,y \in X$ such that $x \neq y$, then $\overline{\{x\}} \neq \overline{\{y\}}$ and there exist disjoint semi open sets containing x and y, respectively, which implies (X,T) is semi-T_2.

Theorem 2.3. *Every semi-*T_2 *space is s-weakly Hausdorff.*

The straightforward proof is omitted.

Combining the results above yield the following corollary.

Corollary 2.1. *A space is s-weakly Hausdorff iff its T_0-identification space is semi-T_2.*

Since the natural map $P:(X,T) \to (X_0,S_0)$ has such strong properties, many questions about s-weakly Hausdorff spaces can be reduced to questions about semi-T_2 spaces.

By Theorem 2.1 (g), if (X,T) is s-weakly Hausdorff, then the continuous, closed, open image (X_0,S_0) is s-weakly Hausdorff. The following example shows that the continuous closed, open image of a s-weakly Hausdorff space need not be s-weakly Hausdorff.

Example 2.1. Let $X=\{a,b,c,d\}$, $Y=\{e,f\}$, $T=\{X,\phi,\{a,b\},\{d\},\{a,b,d\}\}$, $S=\{Y,\phi,\{e\}\}$, and let $f=\{(a,e),(b,e),(d,e),(c,f)\}$. Then f is a continuous, closed, open function from the s-weakly Hausdorff space (X,T) onto the non s-weakly Hausdorff space (Y,S).

Since every open set is an α-set, then α-set in Theorem 2.1 (f) can be replaced by open. The space (X,T) in Example 2.1 can be used to show that α-set in Theorem 2.1 (f) can not be replaced by closed and semi open and to show that the converse of Theorem 2.3 is false.

3. Product Spaces and Other Separation Axioms.

Theorem 3.1. *If (X_α,T_α) is s-weakly Hausdorff for all $\alpha \in A$, then $(\prod_{\alpha\in A} X_\alpha, P)$ is s-weakly Hausdorff, where P denotes the product topology on $X = \prod_{\alpha\in A} X_\alpha$.*

Proof. Let $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Since $\overline{\{x\}} = \prod_{\alpha\in A}\overline{\{x_\alpha\}}$ and $\overline{\{y\}} = \prod_{\alpha\in A}\overline{\{y_\alpha\}}$, then there exists $\beta \in A$ such that $\overline{\{x_\beta\}} \neq \overline{\{y_\beta\}}$. Let U_β and V_β be disjoint semi open sets in X_β containing x_β and y_β, respectively. For each $\alpha \neq \beta$ let

$U_\alpha = X_\alpha = V_\alpha$. Then $\prod_{\alpha\in A} U_\alpha$, $\prod_{\alpha\in A} V_\alpha$ are disjoint semi open sets in X containing x and y, respectively.

The following example shows that the converse of Theorem 3.1 is false.

Example 3.1. Let $X = \{a,b,c\}$, $Y = (0,1)$, $T = \{X,\phi,\{a\},\{a,b\},\{c\},\{a,c\}\}$, and S be the usual topology on Y. Then (X,T) is not s-weakly Hausdorff and $(X \times Y,P)$ is s-weakly Hausdorff.

Theorem 3.2. *Every s-regular space is s-weakly Hausdorff.*

Theorem 3.3. *Every semi-R_1 space is s-weakly Hausdorff.*

The proofs for the two theorems above follow directly from definitions and Theorem 2.1 (c) and are omitted.

Example 3.1 can be used to show that the cconverse of Theorem 3.2 is false. The set $C = \{b\} \times Y$ is closed in $X \times Y$, $x = (a, \frac{1}{2}) \notin C$, and there does not exist disjoint semi open sets containing x and C, respectively.

The following example shows that the converse of Theorem 3.3 is false.

Example 3.2. Let N denote the natural numbers and let T be the usual topology on N. Then $(\beta N,W)$, the Stone - Cech compactification of (N,T), is extremely disconnected and contains non isolated points [11]. Let x be a non isolated point of βN and let $y \notin \beta N$. Then

$S = \{O \in W \mid x \notin O\} \cup \{O \cup \{y\} \mid x \in O \in W\}$ is a topology on

$Y = \beta N \cup \{y\}$ and (Y,S) is regular, which implies (Y,S) is s-weakly Hausdorff. Since $scl\{x\} = \{x\} \neq \{y\} = scl\{y\}$ and there does not exist disjoint semi open sets containing x and y, respectively, then (Y,S) is not semi-R_1.

Example 3.2 shows that T_0 in Theorem 2.2 can not be replaced by semi-T_0 or semi-T_1.

The next theorem follows directly from definitions and the fact that every semi convergent net is convergent.

Theorem 3.4. *Every weakly Hausdorff space is s-weakly Hausdorff.*

Since every T_2 space is weakly Hausdorff, then every T_2 space is s-weakly Hausdorff. The space (X,T) in Example 2.1 shows that a s-weakly Hausdorff space need not be T_2 or weakly Hausdorff.

Theorem 3.5. *Every s-weakly Hausdorff space is semi-R_0.*

The straightforward proof is omitted.

The next example shows that the converse of Theorem 3.5 is false.

Example 3.3. Let X be an infinite set and let T be the finite complement topology on x. Then (X,T) is semi-R_0 and not s-weakly Hausdorff.

The space (X,T) in Example 2.1 can bec combined with Example 3.3 to show that each of T_i and semi-T_i, i=0,1, is independent of s-weakly Hausdorff.

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