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ON A RIEMANNIAN MANIFOLD M_n WITH STRUCTURES DEFINED BY NILPOTENT OPERATORS

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In the last fifteen years a study of differentiable structures defined by nilpotent operators of any finite degree was made by H.A. Eliopoulos [2], [3], [4], [5] and by M^me Lehmann-Lejeune [7]. The structures in [2] and [3] are the Almost tangent structured. C.S. Houb connected in [6] the existence of Almost tangent structures with the existence of Almost product and Almost complex structures on manifolds M_{2n} of even dimension. In this note we consider a Riemannian manifold M_n on which an r-tangent structure J has been defined such that $r+1=2\varkappa$. We are able to connect such a structure J with an Almost complex and an Almost product structure.

1. Let M_n be a Riemannian manifold of class C^∞. Let T_x be the tangent space at the point x of V_n and T_x^c the complexification of T_x. We assume that a field of class C^∞ of linear operators J_x is given on M_n satisfying:

a) At each point $x \in M_n$, J_x maps the space T_x^c in to T_x^c

b) There exists an integer $r > 0$ such that

$$J_x^r \neq 0 \ , \ J_x^{r+1} = 0 \ , \ \forall x \in V_n$$

c) $\dim(I_m J_x^r) = \dim(\mathrm{Ker}\ J_x) = q$.

d) $r+1 = 2\varkappa$.

Proposition 1.1. *There exist distributions* F_x^1, $F_x^2, \ldots, F_x^{r+1}$ *of* M_n *of dimension* q *such that:*

a) $T_x^c = F_x^1 + F_x^2 + \ldots + F_x^{r+1}$

b) $J_x(F_x^i) = F_x^{i-1}$, $\forall i \in \{2,3,\ldots,r+1\}$

Proof. Let

$$E_x^i = \mathrm{Ker}\ J_x^i \quad \forall i \in \{0,1,\ldots,r+1\} \tag{1.1}$$

Then [1]

$$\{0\} = E_x^0 \subset E_x^1 \subset \ldots \subset E_x^r \subset E_x^{r+1} = T_x^c$$

We know [1] that there exist $r+1$ subspaces of T_x^c, F_x^1, F_x^2, ..., F_x^{r+1} which satisfy:

$$E_x^i = E_x^{i-1} \oplus F_x^i \quad \text{and} \tag{1.3a}$$

J_x maps F_i into F_{i-1} and this mapping is bijective. (1.3b)

From (1.3a) and (1.3b) we have:

$$T_x^c = E_x^{r+1} \tag{1.4}$$

From (1.3a) and (1.4) we have:

$$T_x^c = E_x^{r+1} = E_x^r \oplus F_x^{r+1} = \ldots = F_x^1 + F_x^2 + \ldots + F_x^{r+1} \tag{1.5}$$

From (1.3b) we have:

$$\dim F_x^i \leq \dim F_x^{i-1}, \ \forall x \in \{2,\ldots,r+1\} \tag{1.6}$$

We know that

$$\dim T_x^c = \dim(\ker J_x^r) + \dim (\mathrm{Im} J_x^r) \text{ , that is}$$

$$\dim \ker(J_x^r) = n-q$$

Or

$$\dim E_x^r = n-q \tag{1.7}$$

From (1.7) follows that

$$\dim F_x^{r+1} = \dim M_n - \dim E_x^r = n-n+q = q \tag{1.8}$$

That is

$$q = \dim F_x^{r+1} \leq \dim F_x^r \leq \ldots \leq \dim F_x^1 = q \tag{1.9}$$

Hence

$$\dim F_x^1 = \dim F_x^2 = \ldots = \dim F_x^{r+1} = q \tag{1.10}$$

From (1.3b) and (1.10) we have

$$J_x(F^i) = F_x^{i-1} \quad \forall\, i \in \{2,3,\ldots,r+1\}$$

We now introduce in the manifold a local coordinate system. Then in any local coordinate neighborhood at the point x , one can write the operator J_x and distributions F_x^i $(i = 1, 2,\ldots,r+1)$ in the tensorial notations:

$$F_i^j \quad (i, j = 1,2,\ldots,n) \tag{1.11}$$

$$B^i_{\alpha_\varkappa} \; i\in\{1,2,\ldots,n\}\alpha_1\{1,2,\ldots,\}\alpha_\varkappa \in \{(\varkappa-1)q,(\varkappa-1)q+1,\ldots.\varkappa q\}$$

One can always arrange that

$$F_j^i \, B^j_{\alpha_i} = B^i_{\alpha_{i-1}} \tag{1.12}$$

where here and in the following Latin indices run over the range $1,2,\ldots,n$ and

$$\alpha_1 \in \{1,2,\ldots,q+1\},\ \alpha_\varkappa \in \{(\varkappa-1)q,(\varkappa-1)+1,\ldots,\varkappa q\}\ ,$$
$$\forall\ \varkappa \in \{2,\ldots,r+1\}\ .$$

The matrix $(B^i_{\alpha_1},B^i_{\alpha_2},\ldots,B^i_{\alpha_{r+1}})$ has an inverse say

$$\begin{pmatrix} {}^{\alpha_1}B_i \\ {}^{\alpha_2}B_i \\ \cdot \\ \cdot \\ {}^{\alpha_{r+1}}B_i \end{pmatrix}$$

Then

$$B^j_{\alpha_\varkappa}\ {}^{\beta_\varkappa}B_j = \delta^\beta_\alpha\ ,\ {}^{\beta_\varkappa}B_j\ B^j_{\alpha_\varkappa} = \delta^\beta_\alpha$$

$$B^j_{\alpha_i}\ {}^{\beta_1}B_j = 0 \text{ for } i \neq 1,\ {}^{\beta_i}B_j\ B^j_{\alpha_1} = 0 \text{ for } i \neq 1 \qquad (1.13)$$

$$B^j_{\alpha_1}\ {}^{\alpha_1}B_i + B^j_{\alpha_2}\ {}^{\alpha_2}B_i + \ldots + B^j_{\alpha_{r+1}}\ {}^{\alpha_{r+1}}B_i = \delta^j_i$$

Here and in the following

$$B^j_{\alpha_1}\ {}^{\alpha_1}B_i = \sum_{\alpha_1=1}^{q} B^j_{\alpha_1}\ {}^{\alpha_1}B_i$$

$$B^j_{\alpha_\varkappa}\ {}^{\alpha_\varkappa}B_i = \sum_{\alpha_\varkappa=(\varkappa-1)q}^{\varkappa q} B^j_{\alpha_\varkappa}\ {}^{\alpha_\varkappa}B_i,\ \forall\ \varkappa \in \{2,\ldots,r+1\}$$

We put

$${}_\varkappa B^j_i = {}^{\alpha_\varkappa}B_i\ B^j_{\alpha_\varkappa}\ ,\ \forall\ \varkappa \in \{1,2,\ldots,r+1\} \qquad (1.14)$$

Then

$$ {}^{1}B_i^j + {}^{2}B_i^j + \ldots + {}^{r+1}B_i^j = \delta_i^j \tag{1.15} $$

The following equalities are easy to verify

$$ \begin{aligned} &{}^{\varkappa}B_i^j \; {}^{1}B_j^h = 0 \quad \text{for} \quad \varkappa \neq 1 \\ &{}^{\varkappa}B_i^j \; {}^{\varkappa}B_j^h = {}^{\varkappa}B_i^h \end{aligned} \tag{1.16} $$

$$ \begin{aligned} &{}^{\varkappa}B_i^h \; B_h^{\alpha_\varkappa} = B_i^{\alpha_\varkappa} \; {}^{\varkappa}B_i^h \; B_h^{\alpha_1} = 0 \quad \text{for} \quad 1 \neq \varkappa \\ &{}^{\varkappa}B_i^h \; B_{\alpha_\varkappa}^i = B_{\alpha_\varkappa}^h \; {}^{\varkappa}B_i^h \; B_{\alpha_1}^i = 0 \quad \text{for} \quad 1 \neq \varkappa \end{aligned} \tag{1.17} $$

Let $J, {}^{1}B, {}^{2}B, \ldots, {}^{r+1}B$ the operators on T_x^c defined by the tensors $F_i^j, {}^{1}B_i^j, \ldots, {}^{r+1}B_i^j$.

Then it is easy to see that ${}^{1}B.J = J$, $J\,{}^{1}B = 0$, ${}^{\varkappa}B.J = 0$, $J.{}^{\varkappa}B = J$, $\forall \varkappa \in \{2,3,\ldots,r+1\}$.

That is

$$ \begin{aligned} &F_\varkappa^j \; {}^{1}B_1^\varkappa = 0 \qquad {}^{1}B_i^j \; F_\varkappa^i = F_\varkappa^j \\ &{}^{\varkappa}B_j^i \; F_r^j = 0 \qquad F_i^j \; {}^{\varkappa}B_r^i = F_r^j \end{aligned} \tag{1.18} $$

2. Let

$$ \begin{aligned} A_{ji} &= B_j^{\alpha_1} \; B_i^{\alpha_1} + B_j^{\alpha_2} \; B_i^{\alpha_2} + \ldots + B_j^{\alpha_{r+1}} \; B_i^{\alpha_{r+1}} = \\ &= \sum_{\alpha_1=1}^{q} B_j^{\alpha_1} \; B_i^{\alpha_1} + \ldots + \sum_{\alpha_{r+1}=rq}^{(r+1)} B_j^{\alpha_{r+1}} \; B_i^{\alpha_{r+1}} \end{aligned} \tag{2.1} $$

The A_{ij} is symmetric and:

$$ \begin{aligned} &A_{ji} \; B_{\alpha_\varkappa}^j \; B_{\beta_\varkappa}^i = \delta_{\alpha\beta}, \quad A_{ji} \; B_{\alpha_\varkappa}^j \; B_{\alpha_\lambda}^i = 0 \qquad \varkappa \neq \lambda \\ &A_{ji} \; B_{\alpha_\varkappa}^i = B_j^{\alpha_\varkappa} \end{aligned} \tag{2.2} $$

These identities show that (A_{ij}) is non-singular and that $(B^i_{\alpha_1}, B^i_{\alpha_2}, \ldots, B^i_{\alpha_{r+1}})$ form an orthonormal frame with respect to A_{ij}.

Let

$$ {}^{\varkappa}B_{ji} = {}^{\varkappa}B^h_j \, A_{hi} \ , \quad \forall \varkappa \in \{1,2,\ldots,r+1\}. \tag{2.3} $$

Then

$$ {}^{\varkappa}B_{ji} = B^{\alpha_\varkappa}_j B^{\alpha_\varkappa}_i \ , \quad \forall \varkappa \in \{1,2,\ldots,r+1\}. $$

$$ {}^{1}B_{ji} + {}^{2}B_{ji} + \ldots + {}^{r+1}B_{ji} = A_{ji} \tag{2.4} $$

${}^{1}B_{ji}, {}^{2}B_{ji}, \ldots, {}^{r+1}B_{ji}$ are symmetric and

$$ {}^{\varkappa}B_{ji} = {}^{\varkappa}B^t_j \, {}^{\varkappa}B^s_i \, A_{ts} \tag{2.5} $$

$$ {}^{\varkappa}B^t_j \, {}^{1}B^s_i \, A_{ts} = 0 \quad \text{for} \quad 1 \neq \varkappa $$

We consider now the tensor G_{ij} defined by

$$ G_{ji} = \frac{1}{2} \left(A_{ji} + F^t_j \, F^s_i \, A_{ts} + {}^{1}B_{ji}\right) \tag{2.6} $$

Proposition 2.1. *The tensor G_{ij} satisfies the following identites.*

$$ G_{ji} \, B^i_{\alpha_\varkappa} = B^{\alpha_\varkappa}_j \ , \quad \forall \varkappa \in \{1,2,\ldots,r+1\}. \tag{2.7 A} $$

$$ G_{ti} \, {}^{1}B^t_j = {}^{1}B_{ji} \tag{2.7 B} $$

Proof. A) Let $\varkappa=1$ Then

$$ G_{ji} \, B^i_{\alpha_1} = \frac{1}{2} \left(A_{ji} \, B^i_{\alpha_1} + F^t_j \, F^s_i \, A_{ts} \, B^i_{\alpha_1} + {}^{1}B_{ji} \, B^i_{\alpha_1}\right) $$

Noticing that $A_{ji} \, B^i_{\alpha_1} = B^{\alpha_1}_j$ because of (2.2), $F^S_i \, B^i_{\alpha_1} = 0$

because of $J(F_1) = 0$ and ${}^1B_{ji}\, B^i_{\alpha_1} = B_j^{\alpha_1}$, we have

$$G_{ji}\, B^i_{\alpha_1} = \frac{1}{2}\,(B_j^{\alpha_1} + B_j^{\alpha_1}) = B_j^{\alpha_1}$$

For $\varkappa = 2,3,\ldots,r,r+1$ we have

$$G_{ji}\, B^i_{\alpha_\varkappa} = \frac{1}{2}\,(A_{ji}\, B^i_{\alpha_\varkappa} + F_j^t\, F_i^s\, A_{ts}\, B^i_{\alpha_\varkappa} + {}^1B_{ji}\, B^i_{\alpha_\varkappa})$$

Noticing that

$A_{ji}\, B^i_{\alpha_\varkappa} = B_j^{\alpha_\varkappa}$ because of (2.2), $F_i^s\, B^i_{\alpha_\varkappa} = B^s_{\alpha_{\varkappa-1}}$ and ${}^1B_{ji}\, B^i_{\alpha_\varkappa} = 0$, we have

$$G_{ji}\, B^i_{\alpha_\varkappa} = \frac{1}{2}\,(B_j^{\alpha_\varkappa} + F_j^t\, B^s_{\alpha_{\varkappa-1}}\, A_{ts})$$

But $B^s_{\alpha_{\varkappa-1}}\, A_{ts} = B_t^{\alpha_{\varkappa-1}}$ because of (2.2).

Hence

$$G_{ji}\, B^i_{\alpha_\varkappa} = \frac{1}{2}\,(B_j^{\alpha_\varkappa} + F_j^t\, B_t^{\alpha_{\varkappa-1}}) = \frac{1}{2}\,(B_j^{\alpha_\varkappa} + B_j^{\alpha_\varkappa}) = B_j^{\alpha_\varkappa}$$

B) $G_{ti}\, {}^1B^t_j = \frac{1}{2}\,(A_{ti}\, {}^1B^t_j + F^\varkappa_t\, F^s_i\, A_{\varkappa s}\, {}^1B^t_j + {}^1B_{ti}\, {}^1B^t_j)$

Noticing that $A_{ti}\, {}^1B^t_j = {}^1B_{ij}$ because of (2.3) $F^\varkappa_t\, {}^1B^t_j = 0$ because of (1.17) and ${}^1B_{ti}\, {}^1B^t_j = {}^1B_{ji}$ because of (2.4), we have

$$G_{ti}\, {}^1B^t_j = \frac{1}{2}\,({}^1B_{ji} + {}^1B_{ji}) = {}^1B_{ji}.$$

Relations A of the above proposition show that (G_{ij}) is non-singular. Let (G^{ij}) be the inverse matrix of (G_{ij}), then

$$G^{ij}\, B_j^{\alpha_\varkappa} = B^i_{\alpha_\varkappa}\,, \quad \forall\, \varkappa \in \{1,2,\ldots,r+1\}. \qquad (2.8)$$

Originally $B_i^{\alpha_1}$, $B_i^{\alpha_2},\ldots,B_i^{\alpha_{r+1}}$ were defined in (1.14) from the matrix $(B^i_{\alpha_1}, B^i_{\alpha_2},\ldots,B^i_{\alpha_{r+1}})$, now we have found a Riemannian metric G_{ij} so that they are related by (2.8).

The following equalities are easy to verify:

$$\begin{aligned} &G_{st}\,.\,{}^{\varkappa}B_i^j\,{}^{\varkappa}B_j^t = {}^{\varkappa}B_{ij} \quad \forall\,\varkappa\in\{1,2,\ldots,r+1\} \\ &G_{jt}\,.\,{}^{\varkappa}B_i^j\,{}^{\ell}B_j^t = 0 \quad \text{for} \quad \varkappa\neq\ell \end{aligned} \tag{2.9}$$

3. We consider the operators

$$\begin{aligned} {}^{\lambda}\Psi_{\iota\varkappa} &= B_\iota^{\alpha_\lambda}\,B_\varkappa^{\alpha_{\lambda-1}} + B_\varkappa^{\alpha_\lambda}\,B_\iota^{\alpha_{\lambda-1}} \\ {}^{\lambda}\Phi_{\iota\varkappa} &= B_\iota^{\alpha_\lambda}\,B_\varkappa^{\alpha_{\lambda-1}} - B_\varkappa^{\alpha_\lambda}\,B_\iota^{\alpha_{\lambda-1}} \end{aligned} \tag{3.1}$$

$$\lambda\in\{2,4,\ldots,r-1,\ r+1=2m\}$$

If we define

$$\begin{aligned} {}^{\lambda}\Psi_\iota^\varkappa &= G^{\varkappa j}\,.\,\Psi_{ij} \\ {}^{\lambda}\Phi_\iota^\varkappa &= G^{\varkappa j}\,.\,\Phi_{ij} \end{aligned} \tag{3.2}$$

$$\lambda\in\{2,4,\ldots,r-1,\ r+1=2m\}$$

then we have because of relation (2.8)

$$\begin{aligned} {}^{\lambda}\Psi_\iota^\varkappa &= B_\iota^{\alpha_\lambda}\,B^\varkappa_{\alpha_{\lambda-1}} + B^\varkappa_{\alpha_\lambda}\,B_i^{\alpha_{\lambda-1}} \\ {}^{\lambda}\Phi_\iota^\varkappa &= B_\iota^{\alpha_\lambda}\,B^\varkappa_{\alpha_{\lambda-1}} - B^\varkappa_{\alpha_\lambda}\,B_\iota^{\alpha_{\lambda-1}} \end{aligned} \tag{3.3}$$

Proposition 3.1. *The tensor field* ${}^{\lambda}\Psi_\iota^\varkappa$ *(*$\lambda\in\{2,4,\ldots,r+1\}$*) satisfies*

$$ {}^{\lambda}\psi_{\kappa}^{\iota}\ {}^{\lambda}\psi_{j}^{\kappa} = {}^{\lambda-1}B_{j}^{\iota} + {}^{\lambda}B_{j}^{\iota} $$

Proof.
$$ {}^{\lambda}\psi_{\kappa}^{\iota}\ {}^{\lambda}\psi_{j}^{\kappa} = (B_{\kappa}^{\alpha_\lambda}\ B_{\alpha_{\lambda-1}}^{\iota} + B_{\kappa}^{\alpha_{\lambda-1}}\ B_{\alpha_\lambda}^{\iota}) \cdot (B_{j}^{\beta_\lambda}\ B_{\beta_{\lambda-1}}^{\kappa} + B_{j}^{\beta_{\lambda-1}}\ B_{\beta_\lambda}^{\kappa} $$

Noticing that $B_{\kappa}^{\alpha_\kappa}\ B_{\beta_\ell}^{\kappa} = 0$ for $\kappa \neq \ell$, $B_{\kappa}^{\alpha_\lambda}\ B_{\beta_\lambda}^{\kappa} = \delta_{\beta}^{\alpha}$

we have that

$$ {}^{\lambda}\psi_{\kappa}^{\iota}\ {}^{\lambda}\psi_{j}^{\kappa} = B_{j}^{\alpha_{\lambda-1}}\ B_{\alpha_{\lambda-1}}^{\iota} + B_{i}^{\alpha_\lambda}\ B_{\alpha_\lambda}^{\iota} \quad \text{that is} $$

$$ {}^{\lambda}\psi_{\kappa}^{\iota}\ {}^{\lambda}\psi_{j}^{\kappa} = {}^{\lambda-1}B_{j}^{\iota} + {}^{\lambda}B_{j}^{\iota} $$

In the same way we can prove that following

Proposition 3.2. *The tensor field* ${}^{\lambda}\phi_{\iota}^{\kappa}$ *(*$\lambda \in \{2,4,\ldots, r+1\}$*) satisfies*

$$ {}^{\lambda}\phi_{\kappa}^{\iota}\ {}^{\lambda}\phi_{j}^{\kappa} = -{}^{\lambda-1}B_{j}^{\iota} - {}^{\lambda}B_{j}^{\iota} $$

We also have:

$$ {}^{\lambda}\psi_{\kappa}^{\iota}\ {}^{\lambda}\phi_{j}^{\kappa} = B_{\alpha_\lambda}^{\iota\iota}\ B_{j}^{\alpha_\lambda} - B_{\alpha_{\lambda-1}}^{\iota}\ B_{j}^{\alpha_{\lambda-1}} $$

$$ {}^{\lambda}\phi_{\kappa}^{\iota}\ {}^{\lambda}\psi_{j}^{\kappa} = B_{\alpha_\lambda}^{\iota}\ B_{j}^{\alpha_\lambda} + B_{\alpha_{\lambda-1}}^{\iota}\ B_{j}^{\alpha_{\lambda-1}} $$

Thus (3.4) $\quad {}^{\lambda}\psi\ {}^{\lambda}\phi = -{}^{\lambda}\phi\ {}^{\lambda}\psi$

If we consider now the operators

$$ \psi_{\iota}^{\kappa} = {}^{2}\psi_{\iota}^{\kappa} + {}^{4}\psi_{\iota}^{\kappa} + \ldots + {}^{r+1}\psi_{\iota}^{\kappa} $$

$$ \phi_{\iota}^{\kappa} = {}^{2}\phi_{\iota}^{\kappa} + {}^{4}\phi_{\iota}^{\kappa} + \ldots + {}^{r+1}\phi_{\iota}^{\kappa} \qquad (3.5) $$

we can prove easily that

$$\Psi_{\kappa}^{\lambda}\ \Psi_{j}^{\kappa} = \delta_{j}^{\lambda}$$
$$\Phi_{\kappa}^{\lambda}\ \Phi_{j}^{\kappa} = -\delta_{j}^{\lambda} \qquad (3.6)$$

and

$$\Psi_{\kappa}^{\lambda}\ \Phi_{j}^{\kappa} = -\Phi_{\kappa}^{\lambda}\ \Psi_{j}^{\kappa} \qquad (3.7)$$

Hence we have proved the following theorem:

Theorem 3.1. *If on a Riemannian manifold M_n there is an r-tangent structure J satisfuing $r+1=2k$, then we can always define on M_n an almost product structure Ψ and an almost complex structure Φ which satisfy $\Phi\Psi = -\Psi\Phi$. Hence according to [7] we can define on M_n an almost tangent structure F.*

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