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A CHOQUET ORDERING AND UNIQUE DECOMPOSITIONS IN CONVEX SETS OF TIGHT MEASURES

By
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We introduce a Choquet ordering in the set of representing tight measures on convex sets of finite tight measures and show that for all sets H for which in [7] a Choquet integral representation theorem was proved the following is true: every element of H is represented by a unique maximal probability measure iff H is a simplex.

1. Introduction

This is a continuation of [7]. There we proved for certain convex sets of measures that each element is the barycenter of a probability measure on the extreme points (or "can be decomposed into extremal elements") - an analogue of the Choquet representation theorem. The corresponding uniqueness result was proved under certain metrizability conditions. In the nonmetrizable classical situation uniqueness can not be proved within the set of measures concen -

trated on the extreme points. Several partial orderings are introduced "as means of determining how 'close' to the 'boundary' of the convex set the various measures under consideration live" ([4]). The Choquet-Meyer uniqueness theorem gives conditions equivalent to the uniqueness of representing measures which "live near the extreme points". The latter is made precise by means of the Choquet ordering (for this and other facts from the classical theory we refer to [1] and [5]).

We define a Choquet ordering for convex sets H of measures and prove a uniqueness result of the Choquet-Meyer type: representing maximal probability measures are unique iff the set in question is a simplex. If the elements of H are strictly tight the maximal measures are exactly those measures which are concentrated on the extreme points. Results of this type were already mentioned in [9]. Here they are proved for all sets, in which we can prove the existence of integral representation in [7] and which we hope to be useful in application. We tried to illustrate this in [8],part II by a number of examples.

We need some notation. Let (X,t) be a (not necessarily Hausdorff) topological space with Borel (Baire) - σ-algebra $\underline{B} = \underline{B}(X,t)$ ($\underline{B}_0(X,t)$) and bounded Borel (Baire) measurable functions $B(X,t)$ ($B_0(X,t)$) (we omit the topology if convenient). $P(X,t)$ is the set of tight probability measures on $\underline{B}$ and $M_+(X,t)$ the cone $R_+ \cdot P(X,t)$ (a measure ν on $\underline{B}$ is tight iff $\nu(B) = \sup\{\nu(C): C \subset B \text{ compact}\}$; R_+ contains zero). In the sequel by a measure we always mean a tight measure if nothing else is said. If $M \subset M_+(X,t)$ then the weak topology $\upsilon M(X,t)$ is the weakest topology on M which makes the functional $\nu \to \int_X f \, d\nu$ lower semicontinuous whenever $f:X \to R$ is bounded and lower semicontinuous (cf. [6]). We say that

$p \in P(M,\upsilon M)$ represents $\mu \in M_+(X,t)$ or that μ is the barycenter of p iff

$$\mu(B) = \int_M \nu(B)\, dp(\nu) \quad \text{for every } B \in \underline{B} \qquad (1)$$

and write $\mu = R(p)$. $P_\mu(M,\upsilon M)$ is the set of (tight) probability measures on M representing μ. Formula (1) in fact makes sense because of the following lemma:

Lemma. *The functionals on M of the form $\nu \mapsto \nu(B), B \in \underline{B}$ are nonnegative and measurable w.r.t. $\underline{B}(M,\upsilon M)$.*

Proof: 1. The following standard argument is at hand (cf. [3], §0): Let Y be a set and $\underline{E}$ a collection of subsets closed w.r.t. finite intersection. Let further H be a linear space of functions on Y such that it contains the constant functions, the characteristic functions 1_E for each $E \in \underline{E}$, and $\sup f_n$ for each increasing sequence $(f_n : n \in N)$, provided $\sup f_n$ is bounded. Then: H contains every bounded function which is measurable w.r.t. the σ-algebra generated by $\underline{E}$.

2. Set in 1. $Y = X$, $\underline{E} = t$ and $H = \{h \in B(X,t) : \nu \to \int_X h\, d\nu\}$ is measurable on M w.r.t. $\underline{B}(M,\upsilon M)$. Clearly H contains the constant functions and 1_G for all open G, since it is lower semicontinuous. The last assumption also being fulfilled, we have $B(X,t) \in H$, which proves the lemma.

To motivate our definition of a Choquet ordering in (4) below we consider a special case. Let (X,t) be completely regular (we assume a completely regular space to be Hausdorff by definition). Then $M(X,t) = M_+(X,t) - M_+(X,t)$ endowed with the topology $\sigma(M(X,t), C_b(X,t))$ induced by the set $C_b(X,t)$ of bounded continuous functions on X is a (Hausdorff) locally convex topological vector space and this topology coincides on $M_+(X,t)$ with the weak topology $\upsilon M_+(X,t)$

([6]). Let $M \subset M_+(X,t)$ be convex and bounded (i.e. $\sup\{\nu(X): \nu \in M\} < \infty$). Then in the classical Choquet theory $\mu \in M$ is called the barycenter of $p \in P(M, \upsilon M)$ ($\mu = r(p)$) iff

$$F(\mu) = \int_M F(\nu)\, dp(\nu) \quad \text{for every } F \in M(X,t)'$$

where $M(X,t)'$ is the topological dual of $M(X,t)$. This is equivalent to

$$\mu(f) = \int_M \nu(f)\, dp(\nu) \quad \text{for every } f \in C_b(X,t) \qquad (2)$$

(we write $\nu(f)$ for $\int_X f d\nu$ if $f \in B(X,t)$). The Choquet ordering $<$ on $M_+(M, \upsilon M)$ is defined by

$$q < p \quad \text{iff} \quad \int_M \varphi\, dq \leqq \int_M \varphi\, dp \qquad (3)$$

for every $\varphi: M \to R$ which is bounded convex and continuous. For a class $G \subset B(X,t)$ we write $\overset{\vee}{G}(M)$ for the set of convex functionals on M of the form $\nu \to \max_{i \leqq n} (\nu(f_i)+c_i)$ where $f_i \in G$ and $c_i \in R$. By a simple Hahn-Banach argument every continuous convex and bounded φ on M can be seen to be the upper enveloppe of those functionals from $\overset{\vee}{C}_b(X,t)$ which lie below φ (see [1], p.3; the compactness condition there is superfluous; c.f. further [8], p. 117). Thus if (X,t) is completely regular we have $q < p$ iff (3) holds for every $\varphi \in \overset{\vee}{C}_b(X,t)$. In (2) $C_b(X,t)$ can be replaced by $B(X,t)$ (thus in this special case (1) and (2) are equivalent) and we will show that $\overset{\vee}{C}_b(X,t)$ can be replaced by $\overset{\vee}{B}(X,t)$. As we did in (1) with (2) we carry this definition over to the general case and define the announced ordering: for an arbitrary topological space (X,t) and $q, p \in P_\mu(X,t)$

$$q <_\varphi p \text{ iff } \int_M \varphi\, dq \leqq \int_M \varphi\, dp \text{ for every } \varphi \in \overset{\vee}{B}(X,t) \qquad (4)$$

Because of the lemma this formula makes sense.

Remark. *a. If q and p have barycenters and satisfy formally $q <^{0} p$ then they have obviously the same barycenter. Hence we can extend $<^{0}$ to the set $P^{R}(M, \cup M)$ of all $p \in P(M, \cup M)$ which have a barycenter in M.*

b. If M consists only of probability measures, we can define $<^{0}$ by means of the functions

$$\nu \to \psi(\nu) := \max_{i \leq n} \nu(f_i), \quad f_i \in B(X,t)$$

since $\max_{i \leq n} (\nu(f_i)+c_i) = \max_{i \leq n} \nu(f_i+c_i \cdot 1_X)$.

As mentioned above our definition of barycenter and Choquet ordering coincides with the classical one if (X,t) is completely regular:

Proposition 1. *Let (X,t) be completely regular and $M \subset M_+(X,t)$ convex and bounded. Let $\mu \in M$ and $q,p \in P_\mu(M, \cup M)$. Then*

a. $\mu = R(p)$ *iff* $\mu = r(p)$

b. $q <^{0} p$ *iff* $q < p$

c. $<^{0}$ *is an inductive order on* $P_\mu(M, \cup M)$, *provided M is compact.*

Proof. a. Since all $\nu \in M$ are tight (and hence τ-smooth) and p as well (2) extends to lower semicontinuous bounded functions and by the argument used in the proof of the lemma to bounded Borel measurable functions.

b. For the nontrivial part let $(\bar{X},\bar{t})$ be the Stone-Čech compactification of (X,t) and $\bar{f}$ the unique continuous extension of $f \in C_b(X,t)$ to $\bar{X}$. $\bar{M}$ is the canonical embedding of M into $M_+(\bar{X},\bar{t})$ and $\bar{\mu}$, $\bar{\nu}$, $\bar{q}$, $\bar{p}$ are the measures corresponding to μ, ν, q, p under this embedding. From the right side follows

$$\int_{\bar{M}}^{H} \max_{i \leq n} (\bar{\nu}(g_i)+c_i)d\bar{q}(\bar{\nu}) \leq \int_{\bar{M}}^{H} \max_{i \leq n} (\bar{\nu}(g_i)+c_i)d\bar{p}(\bar{\nu}) \qquad (*)$$

for $g_i \in C_b(\bar{X},\bar{t})$. As $\bar{X}$ is compact, $B_0(\bar{X},\bar{t})$ is the smallest class of functions on $\bar{X}$, containing $C_b(\bar{X},\bar{t})$ and closed under bounded pointwise convergence (c.f. [2], Sect. 56).This argument applied in each index $i \leq n$ separately yields (*) for $g_i \in B_0(\bar{X},\bar{t})$.

Again because $\bar{X}$ is compact by [2], Sect 58, Theorem 1, for every $g \in B(\bar{X},\bar{t})$ there is $g_0 \in B_0(\bar{X},\bar{t})$ such that $\int_{\bar{X}} |g-g_0| d\bar{\mu} = 0$. Hence by (1)

$$\bar{p}(\{\bar{\nu} \in \bar{M}: \bar{\nu}(g) \neq \bar{\nu}(g_0)\}) = \bar{q}(\{\bar{\nu} \in \bar{M}: \bar{\nu}(g) \neq \bar{\nu}(g_0)\}) = 0.$$

Thus in (*) the Baire functions can be replaced by Borel functions. As X and M are homeomorphic to their embeddings we can cancel the bars and b. is proved.

c. If M is compact $<$ is known to be an inductive order ([1]), hence $<^{\emptyset}$ by b.

If (X,t) is completely regular, we are in the situation of classical Choquet theory as we mentioned above. If moreover the set M in question is convex and compact the classical Choquet-Meyer existence and uniqueness theorem ([1] and [5]) holds and can be carried over to our situation by proposition 1. But results of the Choquet-Meyer type can be proven also in the more general setting of [7] and [8]. We now formulate these results - already announced in the introduction - for arbitrary topological spaces (X,t).

B. The Main Results.

From now (X,t) is an arbitrary topological space. Let F be a family of (not necessarily bounded) measurable functions on X and s the initial topology of F on X. By $s \vee t$ we denote the topology generated by s and t. We define.

$$\bar{F} := \bigcap_{f \in \mathrm{span}(\bigcup F \cup \{1_X\})} \{g \in C(X, s\vee t) : |g| \leq |f|\}$$

and for $M \subset M_+(X,t)$

$$M^F := \{\nu \in M : L^1(\nu) \supset F\}$$

τM^F is the topology on M^F generated by υM^F and the functionals $\nu \to \int_X g \, d\nu$, $g \in \bar{F}$. Recall that a set K in a linear space V is a simplex iff the cone $R_+(K \times \{1\})$ is a lattice in its own order in $V \times R([1])$.

Theorem. *Let F be a countable family of Borel measurable functions on X. Let H be a convex subset of $M_+^F(X,t)$ which is bounded and closed with respect to $\tau M_+^F(X,t)$. Then:*

1. The relation $\prec_\delta$ is an inductive ordering on $P^R(H, \upsilon H)$. Especially for every μ there is a maximal $p \in P^R(H, \upsilon H)$ with $\varepsilon_\mu \prec_\delta p$ (thus representing μ).

2. The measures in 1. are unique iff H is a simplex.

The proves for this part are given in section C.

To connect the results above with those from [7] we show that every maximal measure is concentrated on the extreme points in some sense. This result gives some more information about representing measures as the corresponding one in [7]. Let for $M \subset M_+(X,t)$ be $\Sigma(M)$ the σ-algebra on M generated by the functionals $\nu \to \nu(B)$, $B \in \mathcal{B}$. If M is convex ex$M$ is the set of extreme points.

Proposition 2. *Let H be as in the theorem and $\mu \in H$. Then for every maximal probability measure p on H representing μ the outer measure of ex H w.r.t. $p|\Sigma(H)$ is one.*

Remark. *a. From this follows [7], Theorem 1, i.e. for every $\mu \in H$ there is a probability measure $\bar{p}$ on Σ(ex H) representing μ ((1) makes sense for such measures).*

b. We do not know if the converse is true: does every probability measure $\bar{p}$ on $\Sigma(H)$ which gives outer measure one to ex H extend to a unique maximal probability measure? So we cannot (in the general case) prove a (desired) uniqueness result corresponding to a., i.e. for measures concentrated on the extreme points. Neither don't we know if there is a suitable σ-algebra $\Sigma'(H)$ with $\Sigma(H) \subset \Sigma'(H) \subset \underline{B}(H, \upsilon H(X,t))$ for which this could be proved.

These difficulties are absent, if the measures in H are strictly tight, i.e. if they are inner regular w.r.t. metrizable compact sets. Then:

Proposition 3. *Let $\mu \in H$ be strictly regular. Then $p \in P_\mu(H, \upsilon H)$ is maximal iff there is a Borel set $B_\mu \subset ex\ H$ such that $p(B_\mu)=1$.*

Corollary. *Let every $\mu \in H$ be strictly regular. Then H is a simplex iff for every $\mu \in H$ there is a unique tight probability measure on $\underline{B}(ex\ H, \upsilon\ ex\ H)$ which represents μ.*

C. The Proofs.

We use a technique already developed in [7]. It is based on the fact that the positive cone generated by H is well-capped and uses special properties of the caps corresponding to the elements of H. We collect these facts in the following lemma. It consists of points a.-l.. The respective proofs are interspersed where convenient. We refer to [7] where possible.

Lemma. *Let F be a countable family of measurable functions on X and let for $M \subset M_+^F(X,t)$ be $\upsilon\sigma M(X,t)$ the topology on M generated by υM and the functionals $\nu \to \int f\, d\nu$,*

$f \in F$. Let further H be a convex bounded subset of $M_+^F(X,t)$ which is closed in $\upsilon\sigma M_+^F(X,t)$ and $\mu \in H$ fixed. Then there is an increasing sequence of compact sets $C_n \subset X$, $n \in N$, such that with $C := \bigcup_{n \in N} C_n$ and $t^0 := r \vee t$ where r is the initial topology of $\{1_{C_n} : n \in N\}$ the following holds:

a. For each $f \in F$ the restriction $f|C$ is t^0-continuous.

b. $(C, C \cap t^0)$ is completely regular.

c. $\underline{B}(X,t) = \underline{B}(X,t^0)$

d. $M_+^C(X,t) := \{\nu \in M_+(X,t) : \nu(X \setminus C) = 0\} =$
$= \{\nu \in M_+(X,t^0) : \nu(X \setminus C) = 0\}$

Proof of a.-d. We use the compact sets C_n constructed in [7], p. 25. As μ is tight they can be chosen such that each restriction $f|C_n$, $f \in F$, $n \in N$ is t-continuous by Lusin's theorem. Because of the definition of t^0 a. holds. As each C_n is t^0-clopen, $(C, C \cap t^0)$ is even locally compact and σ-compact, hence completely regular. c. and d. follow from the lemma in [7], p. 24.

Furthermore the C_n can be taken in such a way that there is a functional $L: M_+(X,t^0) \to [0,\infty]$ of the form $L(\nu) = \int_X g \, d\nu$, where g is a nonnegative $\underline{B}$-measurable function on X, such that

e. $L(\mu) = 1$, $Y^\mu := \{\nu \in R^+ \cdot H : L(\nu) \leq 1\}$ and $(R_+ \cdot H) \setminus Y^\mu$ are convex.

f. For every $\nu \in Y^\mu$ the set $X \setminus C$ is ν-null.

g. Y^μ is compact in the locally convex topology $\upsilon M_+^C(C, C \cap t^0)$ generated on $M_+^C(X,t)$ by the functionals $\nu \to \int_C f \, d\nu$, $f \in C_b(C, C \cap t^0)$.

Proof of e.-g. The construction of the cap Y^μ was carried out in [7], p. 26 and yields f. 5) and 6) there guarantee the compactness in g. The rest of g. follows from

B. and f..

We show now that the probability measures on Y^μ tight with respect to the weak topologies coming from t and t^0 respectively are essentially the same. To this end we examine the Baire sets.

h. $\underline{B}_0(Y^\mu, \upsilon Y^\mu(X,t)) \subset \underline{B}_0(Y^\mu, \upsilon Y^\mu(C, C\cap t^0)) = \underline{A}$

where $\underline{A}$ *is the* σ*-algebra on* Y^μ *generated by the functionals* $\nu \to \int_C f\, d\nu$, $f \in C_b(C, C\cap t^0)$.

Proof. The inclusion holds since for each t-lower semicontinuous function f on X the function $f|C$ is t^0-lower semicontinuous on C and induces the same evaluation on Y^μ because of f.. Hence the topology $\upsilon Y^\mu(X,t)$ is smaller than $\upsilon Y^\mu(C, C\cap t^0)$. To prove the equation we note that the set of polynomials

$$\nu \to \prod_{j=0}^{n} a_j \prod_{\substack{i=1,\dots,j \\ k_1+\dots+k_j=j \\ k_i\in N_0}} (\nu(f_i))^{k_i}, \quad a_j \in R,\ f_i \in C_b(X,t)$$

is uniformly dense in the set of bounded $\upsilon Y^\mu(C, C\cap t^0)$-continuous functionals on Y^μ by the Stone-Weierstraß theorem. Thus they generate the same σ-algebra, i.e. the Baire sets. As they generate $\underline{A}$ as well the equation holds.

i. L is measure affine, i.e. for $\nu \in Y^\mu$ *and a measure p on* Y^μ *representing* ν *we have* $L(\nu) = \int L\, dp$.

Proof. i. is an immediate consequence of the form $L(\nu) = \int_X g\, d\nu$ with a nonnegative measurable function $g: X \to R$ and the monotone convergence theorem.

j. Let $Z^\mu := \{\nu \in Y^\mu : L(\nu) = 1\}$. *If a measure p on* Y^μ *represents a* $\nu \in Z^\mu$ *then* $p(Y^\mu \setminus Z^\mu) = 0$.

Proof. Apply i..

Now we can establish the desired correspondence. Let $p \in P(Y^\mu, \upsilon Y^\mu(X,t))$. As $\underline{A} \subset \underline{B}$ $(Y^\mu, \upsilon Y^\mu(X,t))$ we can restrict p on $\underline{A}$, i.e. the Baire sets of Y^μ w.r.t. $\upsilon Y^\mu(C, C \cap t^0)$(h.). In this topology Y^μ is compact (g.) thus there exists the unique extension $p' \in P(Y^\mu, Y^\mu(C \cap t^0))$ of $p|\underline{A}$. As Y^μ is also compact in the weaker topology $\upsilon Y^\mu(X,t)$ there is no measure different from p with the same restriction on $\underline{A}$. Hence the mapping $p \to p'$ is bijective. For $\nu \in Z^\mu$ we can identify $P_\nu(Z^\mu, \upsilon Z^\mu(X,t))$ and $P_\nu(Y^\mu, \upsilon Y^\mu(X,t))$ because of j.. For p in this set let $\Phi\, p = p'$ with p' as just constructed. By c. and f. Φ preserves the ordering $<$. So:

k. Let $\nu \in Z^\mu$ Φ: $P_\nu(Z^\mu, H\upsilon Z^\mu(X,t)) \to P_\nu(Y^\mu, \upsilon Y^\mu(C, C \cap t^0))$, $p \to p'$ *is bijective and* $q < p$ *iff* $\Phi\, p < \Phi\, p$.

Last we note:

l. If μ *is strictly regular, then* Y^μ *is metrizable.*

Proof. [7], p. 25.

Proof of the theorem. Before we go into the details, we reduce the problem to a simpler one in two steps:

First we may assume H to be closed in the topology $\upsilon\sigma M_+^F(X,t)$ for a suitably chosen countable family F of measurable functions on X by the argument in [7], p. 25 (this topology is defined in the above lemma). Hence the lemma can be applied.

Secondly we can assume that $H \subset P(X,t)$. Otherwise we consider a set H' of probability measures instead of H (like in [7], p. 27):

Choose $\omega \notin X$. $X' := X \cup \{\omega\}$ is indowed with the topology t', which coincides with t on X and has $\{\omega\}$ as a clopen set. Set $\gamma := \sup\{\nu(X): \nu \in H\}$ and for $\nu \in H$ define $W(\nu) \in P(X', t')$ by

$$W(\nu)(B) := \frac{1}{\gamma}\,(\nu(B\cap X) + (\gamma-\nu(X))\,1_B(\omega)) \text{ for } B\in \underline{B}\,(X',t').$$

For a measurable function g on X let be $g' := g$ on X and $g'(\omega) := 0$; $F' := \{f' : f\in F\}$. Then W is an affine bijection of H onto $H' := W(H)$. H' is closed in $M_+^{F'}(X',t')$ with respect to $\upsilon\sigma M_+^{F'}(X',t')$; W is a homoemorphism with respect to the weak and the "$\upsilon\sigma$-"topologies; and ν is the barycenter of p iff $W(\nu)$ is the barycenter of $p_0\,W^{-1}$.

As for $\nu\in H$, $f'\in B(X',t')$ and $c\in R$ the identity

$$W(\nu)(f')+c = \nu\left(\frac{1}{\gamma}\,(f'|X-f'(\omega)\,1_X)\right)+\gamma\,cf'(\omega)$$

holds, the left side is of the form $\nu(g)+d$, $g\in B(X,t)$, $d\in R$. Thus $q <_\sigma p$ iff $q_0W^{-1} < p_0\,W^{-1}$ and the ordering is preserved as well.

In the sequel we assume these two steps to be done. Let $\mu\in H$, L the measure affine mapping and Y ($=Y^\mu$) the cap defined by L from the lemma. Let $H_L := \{\nu\in H: L(\nu) < \infty\}$. H_L generates the same cone as Y (or Z: $= Z^\mu$). If $p\in P(H,\upsilon H)$ has a resultant -say ν- in H_L, then p is concentrated on H_L, since from i. we have $\infty > L(\nu) = \int L\,dp$ and thus $p(\{p\in H: L(\rho) = \infty\}) = 0$. Hence: There is a bijective mapping from $K := R^+\cdot \bigcup_{\nu\in H_L} P_\nu(H,\upsilon H(X,t))$ onto $M_+(Z,\upsilon Z(X,t))$ (the resultant of a finite measure is defined by (1)) given by the formulas:

$$\int_Z h(\nu')dp'(\nu') = \int_H h\left(\frac{\nu}{L(\nu)}\right) L(\nu)\,dp(\nu)$$

$$\int_H g(\nu)dp(\nu) = \int_Z g\left(\frac{\nu'}{\nu'(X)}\right)\nu'(X)dp'(\nu')$$

with g and h nonnegative Borel measurable functions on $R_+\cdot H_L$, $p\in K$ and $p'\in M_+(Z,\upsilon Z(X,t))$. The proof is like that in [8], p. 122. If the inequalities

$$\int_H \varphi(\nu)\, dq(\nu) \leq \int_H \varphi(\nu)\, dp(\nu) \qquad (4)$$

hold for $\varphi \in \overset{\vee}{B}$, then also for φ of the type

$$\varphi(\nu) = \max_{i\leq n} (\nu(f_i)+c_i L(\nu)) = \max_{i\leq n} \nu(f_i+c_i g)$$

Because of

$$\int_Z \max_{i\leq n} (\nu'(f_i)+c_i)\, dp'(\nu') = \int_H \max_{i\leq n} (\nu(f_i)+c_i L(\nu))dp$$

and the analogue equality for q we see that if q and p satisfy (4) for $\varphi \in \overset{\vee}{B}$ then also q' and p'. The reverse direction follows similarly. The mapping Φ': $\{p \in P^R(H,\upsilon H(X,t)): R(p) \in H_L\} \to P(Z,\upsilon Z(X,t)), p \to \frac{1}{L(R(p))}\, p'$ is bijective and preserves the order; further $R(\Phi' p) = \frac{1}{L(R(p))}\, R(p)$.

Ψ: $\Phi \circ \Phi'$ gives us a bijective order preserving mapping from $P_\nu(H,\upsilon H(X,t))$ onto $P_{\nu'}(Y,\upsilon Y\ (C,\ C\cap t^0))$, $\nu' := \frac{\nu}{L(\nu)} \in Z$, for all $\nu \in H_L$, hence especially for μ. This together with Proposition 1 implies the first part of 1. in the theorem. The rest of 1. is straightforward.

Now we assume that H is a simplex. As the elements of H are probability measures this is equivalent with R^+. H being a lattice in its own order. Then the cap Y is a simplex (we refer to [5], p.93; the proof there is purely algebraic and does not depend on special topological assumptions). By g.the set Y is convex and compact in a locally convex topology, thus $P_\mu(C,\ C\cap t^0))$ contains exactly one element p' maximal w.r.t. the ordering $<$, hence w.r.t. $<^0$. $\Psi^{-1} p'$ is the unique $<^0$-maximal probability measure on H representing μ.

Conversely let each $\nu \in H$ have a unique maximal representing probability measure on H. Via Ψ we see that the same is true on Z, hence on Y. Thus Y is a simplex. Now, every element

of H is contained in a cap which is a simplex. We learn again from [5], p. 93 that then $R^+.H$ is a lattice cone in its own order, which means that H is a simplex itself. This completes the proof of the theorem.

Proof of proposition 2. Ψ_p is maximal in $P_\mu(Y, \upsilon Y(C, C \cap t^0))$. In [9] we proved that for this case the proposition is true. The rest is straightforward.

Proof of proposition 3. According to 1. Y is metrizable in $\upsilon Y(C, C \cap t^0)$, hence $exZ = ex\, Y \setminus \{0\}$ a G_δ-set([1], p. 34). A probability measure is maximal iff it is concentrated on this set. $p \in P_\mu(H, \upsilon H(X,t))$ is maximal iff Ψp is maximal. Hence p is maximal iff it is concentrated on the inverse image of exZ under normalization.

Proof of the corollary. Immediate consequence of proposition 3 and the uniqueness result.

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