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SOME FIXED POINT THEOREMS

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1. Introduction.

Let R_+ be the set of nonnegative real numbers and N the set of positive integers.

Recently, Ray and Chatterjee [7] extended the well-known Banach contraction mapping principle and proved some fixed point theorems with the continuous mappings S and T on a metric space X which satisfy a condition of the type

$$F(S^p x, T^q y) < a_1 F(x,y)+a_2 F(x,S^p x)+a_3 F(y,T^q y)$$

for all x,y in X with $x \neq y$, where $p, q \in N$ and $a_i \in R_+$ for i=1,2,3 with $a_1+a_2+a_3 < 1$ and F is a continuous symmetric mapping of $X \times X$ into R_+ such that $F(x,x) = 0$ for all x in X.

The main purpose of this paper is to extend their results to a more general case. For other related results, we

refer to Browder and Petryshyn [1], Edelstein [3], Petryshyn and Williamson [6], Ray [8] and the present author [9].

2. Main Results.

Before going into our discussion, we state some definitions.

Definition 1. (Ciric [2]). Let T be a selfmapping of a metric space X. We say that T is orbitally continuous if $\lim_i T^{n_i}x = u \in X$ for some subsequence $\{T^{n_i}x\}_{i=1}^{\infty}$ of $\{T^n x\}_{n=1}^{\infty}$, where $x \in X$, implies $\lim_i TT^{n_i}x = Tu$. A space X is called T-orbitally complete if every Cauchy sequence which is contained in $\{T^n x\}_{n=1}^{\infty}$ for some $x \in X$ converges in X.

Definition 2. (Kuratowski [5]). Let A be a bounded subset of a metric space X. We denote by $\alpha(A)$, the infimun of all $\varepsilon>0$ such that A admits a finite covering consisting of subsets with diameter less than ε.

Definition 3. (Furi and Vignoli [4]). A continuous selfmapping T of a metric space X is called densifying if for every bounded subset A of X such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

Theorem 1. *Let S and T be continuous selfmappings of a Hausdorff space X. Let F be a continuous symmetric mapping of $X \times X$ into R_+ satisfying the following conditions:*

(C_1) $F(x,x)=0$ for all x in X and $F(x,y)\neq 0$ for all x,y in X, $x\neq y$;

(C_2) $F(x,z) \leq F(x,y) + F(y,z)$ for all x,y,z in X;

(C_3) there exist $p,q \in N$ such that

$F(S^p x, T^q y) < h(F(x,y), F(x,S^p x), F(y,T^q y), F(x,T^q y), F(y,S^p x),$
$(F(x,y))^{-1} F(x,S^p x)F(y,T^q y), (F(x,y))^{-1} F(x,T^q y)F(y,S^p x))$

for all $x,y \in X, x \neq y$, *where* $h:(R_+)^7 \to R_+$ *is a nondecreasing in each coordinate variable and satisfies the condition:*

$$g(t) \equiv h(t,t,t,2t,t,t,t) \leq t \quad \text{for each } t > 0. \qquad (1)$$

If for some $x_0 \in X$ *the sequence* $\{x_n\}_{n=1}^{\infty}$ *consisting of distinct points* x_n *has a convergent subsequence, where* $x_{2n+1} = S^p x_{2n}$ *and* $x_{2n+2} = T^q x_{2n+1}$ *for all* $n \in N_0 \equiv N \cup \{0\}$, *then* S *and* T *have a unique common fixed point.*

Proof. Let $F_n = F(x_n, x_{n+1})$ for $n \in N_0$. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C_3), we have

$$F_{2n+1} = F(S^p x_{2n}, T^q x_{2n+1})$$
$$< h(F(x_{2n},x_{2n+1}),F(x_{2n},x_{2n+1}),F(x_{2n+1},x_{2n+1}),F(x_{2n},x_{2n+2}),$$
$$F(x_{2n+1},x_{2n+1}),(F(x_{2n},x_{2n+1}))^{-1}F(x_{2n},x_{2n+1})F(x_{2n+1},x_{2n+2}),$$
$$(F(x_{2n},x_{2n+1}))^{-1}F(x_{2n},x_{2n+2})F(x_{2n+1},x_{2n+1})).$$

If $F_{2n} < F_{2n+1}$, then

$$F_{2n+1} < h(F_{2n+1},F_{2n+1},F_{2n+1},2F_{2n+1},F_{2n+1},F_{2n+1},F_{2n+1}) \leq F_{2n+1}$$

which is a contradiction. Hence $F_{2n} \geq F_{2n+1}$ for all $n \in N_0$. Similarly, we can prove that $F_{2n+1} \geq F_{2n+2}$ for all $n \in N_0$. Thus $\{F_n\}_{n=0}^{\infty}$ is decreasing and hence converges to some real number $w \in R_+$. Since $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ in X which converges to some $v \in X$, we may put

$v = \lim_k x_{2n_k}$.

Since F and S are continuous,

$$F(v,S^pv) = F(\lim_k x_{2n_k}, S^p\lim_k x_{2n_k}) \tag{2}$$

$$= F(\lim_k x_{2n_k}, \lim_k x_{2n_k+1}) = \lim_k F(x_{2n_k}, x_{2n_k+1})$$

$$= w = \lim_k F(x_{2n_k+1}, x_{2n_k+2}) = F(S^pv, T^qS^pv).$$

If $v \neq S^pv$, then it follows from (2) and (C_3) that

$$F(S^pv,T^qS^pv) < h(F(v,S^pv),F(v,S^pv),F(S^pv,T^qS^pv),F(v,T^qS^pv),$$

$$F(S^pv,S^pv),(F(v,S^pv))^{-1}F(v,S^pv)F(S^pv,T^qS^pv),$$

$$(F(v, S^pv))^{-1}F(v, T^qS^pv)F(S^pv, S^pv))$$

$$\leq g(F(S^pv, T^qS^pv)) \leq F(S^pv, T^qS^pv)$$

which is a contradiction. Thus $v = S^pv$. Similarly, $v=T^qv$.

If u ($\neq v$) were another common fixed point of S^p and T^q, then

$$F(u,v) = F(S^pu, T^qv)$$

$$< h(F(u,v),F(u,u),F(v,v),F(u,v),F(v,u),0,F(u,v))$$

$$\leq g(F(u,v)) \leq F(u,v)$$

which is a contradiction. Hence v is the unique common fixed point of S^p and T^q.

We now show that v is a common fixed point of S and T. Since $S^pSv = SS^pv = Sv$, Sv is also a fixed point of S. By the uniqueness of v, it follows that $Sv = v$. Similarly, $Tv=v$.

Suppose z ($\neq v$) is another common fixed point of S and T. Then

$$F(z,v) = F(S^p z,\ T^q v)$$

$$< h(F(z,v),F(z,z),F(v,v),F(z,v),F(v,z),0,F(v,z))$$

$$\leq g(F(z,v)) \leq F(z,v)$$

which is a contradiction. Hence $z = v$. Thus our proof is complete.

Remark 1. *In Theorem 1, if "$<$" in (C_3) is interchanged with "$\leq$" in (1), then the result of Theorem 1 is still true.*

Corollary 1. *Let S and T be continuous selfmappings of a Hausdorff space X. Let F be a continuous symmetric mapping of $X\times X$ into R_+ satisfying (C_1), (C_2) and*

(C_4) there exists $p,\ q \in N$ such that

$$F(S^p x, T^q y) < a_1(x,y)F(x,y) + a_2(x,y)F(x,S^p x) + a_3(x,y)F(y,T^q y)$$
$$+ a_4(x,y)[F(x,T^q y) + F(y,S^p x)]$$
$$+ a_5(x,y)(F(x,y))^{-1}F(x,S^p x)F(y,T^q y)$$
$$+ a_6(x,y)(F(x,y))^{-1}F(x,T^q y)F(y,S^p x)$$

for all x,y in $X, x \neq y$, where the mappings $a_i : X\times X \to [0,1]$ with the property

$$K = \sup_{x,y\in X} \{a_4(x,y) + \sum_{i=1}^{6} a_i(x,y)\} \leq 1 \qquad (3)$$

If for some $x_0 \in X$ the sequence $\{x_n\}_{n=0}^{\infty}$ consisting of distinct points x_n has a convergent subsequence, where $x_{2n+1} = S^p x_{2n}$ and $x_{2n+2} = T^q x_{2n+1}$ for all $n \in N_0$, then S and T have a unique common fixed point.

Proof. Since

$$a_1 t_1 + a_2 t_2 + a_3 t_3 + \frac{a_4}{2}(t_4 + t_5) + a_5 t_6 + a_6 t_7$$

$$\leq (a_4 + \sum_{i=1}^{6} a_i) \max\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}, t_6, t_7\},$$

we take

$$h(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = K \max\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}, t_6, t_7\}.$$

We can easily see that h satisfies all the conditions of Theorem 1. Hence S and T have a unique common fixed point.

Remark 2. *In Corollary 1, if "$<$" in (C_4) is interchanged with "$\leq$" in (3), then the result of Corollary 1 is still true.*

Remark 3. *Taking $a_1(x,y)=1$ and $a_i(x,y)=0$ for $i=2, 3, \ldots, 6$ and $p=q=1$ in Corollary 1, we obtain a result of Ray [8, Theorem 1].*

Remark 4. *Let $a_i(x,y) = c_i \in [0,1]$ be constants for $i=1,2,3$ and $a_4=a_5=a_6=0$. Then we have a result of Ray and Chatterjee [7, Theorem 1].*

Remark 5. *Let X be a metric space with metric $d=F$. Then Theorem 1 and Corollary 1 hold in metric spaces.*

Remark 6. *Let the space X of Corollary 1 be a metric space with metric d. Taking $p=q=1$, $S=T$, $F=d$, $a_1(x,y)=1$ and $a_i=0$ for $i=2,\ldots,6$, we obtain a result of Edelstein [3, Theorem 1].*

Theorem 2. *Let T be an orbitally continuous self-mapping of a Hausdorff space X. Let F be a continuous symmetric mapping of $X \times X$ into R_+ satisfying (C_1), (C_2) and (C_3) with $p=q=1$ and $S=T$. If for some $x_0 \in X$ the sequence $\{T^n x_0\}_{n=0}^{\infty}$ has a cluster point u in X, then u is a fixed point of T and $\lim_n T^n x_0 = u$.*

Proof. Let $x_n = T^n x_0$ and $F_n = F(x_n, x_{n+1})$ for all $n \in N_0$. If $x_m = x_{m-1}$ for some m, then $u = x_{m-1}$ and the assertion follows. Assume that $x_n \neq x_{n-1}$ for all $n \in N$ and let $\lim_n T^n x_0 = u$. Then as in the proof of Theorem 1, we see that $\{F_n\}_{n=0}^{\infty}$ is decreasing. Since $\lim_i T^{n_i} x_0 = u$ and T is orbitally continuous, it follows that $Tu = \lim_i T^{n_i+1} x_0$, $T^2u = \lim_i T^{n_i+2} x_0$ and

$$\lim_i F(T^{n_i} x_0, T^{n_i+1} x_0) = F(u, Tu), \tag{4}$$

$$\lim_i F(T^{n_i+1} x_0, T^{n_i+2} x_0) = F(Tu. T^2u). \tag{5}$$

Since $\{F_n\}_{n=0}^{\infty}$ is convergent sequence and $\{F_{n_i}\}_{i=0}^{\infty}$ and $\{F_{n_i+1}\}_{i=0}^{\infty}$ are its subsequence, it follows from (4) and (5) that

$$\lim_n F(T^n x_0, T^{n+1} x_0) = F(u, Tu) = F(Tu, T^2u).$$

If $u \neq Tu$, then, by (C_3),

$$\begin{aligned} F(u,Tu) = F(Tu,T^2u) &< h(F(u,Tu), F(u,Tu), F(Tu,T^2u), F(u,T^2u), F(Tu,Tu), \\ &\quad (F(u,Tu))^{-1} F(u,Tu)F(Tu,T^2u), (F(u,Tu))^{-1} F(u,T^2u)F(Tu,Tu)) \\ &\leq g(F(u, Tu)) \leq F(u, Tu) \end{aligned}$$

which is a contradiction. This proves that $Tu = u$.

Corollary 2. *Let X be a metric space and $T: X \to X$ be densifying. Let F be a continuous symmetric mapping of $X \times X$ into R_+ satisfying (C_1), (C_2) and (C_3) with $p=q=1$ and $S = T$. If X is T-orbitally complete and $\{T^n x_0\}_{n=0}^{\infty}$ is bounded for some x_0 in X, then T has a fixed point.*

Proof. Let $A = \bigcup_{n=0}^{\infty} T^n x_0$ ($T^0 x_0 = x_0$). Then $T(A) = \bigcup_{n=1}^{\infty} T^n x_0 \subset A$. Since T is continuous, $T(\bar{A}) \subset \overline{T(A)} \subset \bar{A}$.

To prove $\alpha(A) = 0$, suppose $\alpha(A) > 0$. Then we have

$$\alpha(A) = \max\{\alpha(T(A)), \alpha(x_0)\} = \alpha(T(A))$$

which is a contradiction, since T is densifying. Thus $\alpha(A)=0$ and hence $\alpha(\bar{A})=0$. Then A is totally bounded. Since X is orbitally complete, $\bar{A}$ is compact. Thus, by Theorem 2, T has a fixed point.

3. More Results.

Following Browder and Petryshyn [1], a selfmapping T of a normed linear space X is called asymptotically regular at $x \in X$ if $\|T^n x - T^{n+1} x\| \to 0$ as $n \to \infty$, whenever $T^n x$ is defined for all $n \in N$. The set F(T) denotes the set of fixed points of T in X.

The following theorem generalizes Theorem 3.1 of Ray and Chatterjee [7].

Theorem 3. *Let K be a subset of a uniformly convex Banach space X. Suppose that T is a selfmapping of K such that $F(T) \neq \emptyset$ and*

$$(C) \quad \|Tx-Ty\| < h(\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|, \|x-y\|^{-1}\|x-Tx\|\|y-Ty\|, \|x-y\|^{-1}\|x-Ty\|\|y-Tx\|)$$

for all $x,y \in K$, $x \neq y$, where h is nondecreasing in each coordinate variable and satisfies the condition

$$h(t,at,bt,t,t,t,t) \leq t \quad \text{for each } t>0 \text{ and } a+b \leq 3$$

If there are $x_0 \in K$ and $a_i \in R_+$ for $i=0,1,\ldots,m$, with $\sum_{i=0}^{m} a_i = 1$

such that $S^n x_0$ is defined and lies in K for each $n \in N$, where $S:K \to K$ is the mapping $S = \sum_{i=0}^{m} a_i T^i$, $T^0 = I$ (the identity mapping), then $\|S^n x_0 - S^{n+1} x_0\| \to 0$ as $n \to \infty$, i.e., S is asymptotically regular at x_0.

Proof. Let $p \in F(T)$, $x_n = S^n x_0$ and $d_i = \|T^i x_n - p\|$ for $n \in N_0$ and $i=0,1,\ldots,m$. It is obvious that $F(T) = F(S) \neq \emptyset$. If $d_j = 0$ for some $j=0,1,\ldots,m$, then $d_i = 0$ for all $i=j+1,\ldots,m$. Hence we may assume that $d_i \neq 0$ for all $i \in N_0$. From (C_5)

$$d_i = \|T^i x_n - p\| = \|T^i x_n - T^i p\|$$

$$< h(\|T^{i-1} x_n - p\|, \|T^i x_n - T^{i-1} x_n\|, 0, \|T^{i-1} x_n - p\|, \|p - T^i x_n\|,$$

$$0, \|T^{i-1} x_n - p\|^{-1} \|T^{i-1} x_n - p\| \|p - T^i x_n\|)$$

$$= h(d_{i-1}, d_{i-1} + d_i, 0, d_{i-1}, d_i, 0, d_i).$$

If $d_{i-1} < d_i$, then

$$d_i < h(d_i, 2d_i, d_i, d_i, d_i, d_i, d_i) \leq d_i$$

which is a contradiction. Hence $d_{i-1} \geq d_i$ and $d_i \leq d_0 = \|x_n - p\|$ for $i=1,2,\ldots,m$. Since $\sum_{i=0}^{m} a_i = 1$,

$$\|x_{n+1} - p\| = \|Sx_n - p\| = \left\| \sum_{i=0}^{m} a_i T^i x_n - \sum_{i=0}^{m} a_i p \right\|$$

$$\leq \sum_{i=0}^{m} a_i \|T^i x_n - p\| \leq \sum_{i=0}^{m} a_i \|x_n - p\| = \|x_n - p\|.$$

Hence $\{\|x_n - p\|\}_{n=0}^{\infty}$ is decreasing and $\|x_n - p\| \to r$ for some $r \geq 0$.

If $r=0$, then $x_n \to p$ as $n \to \infty$. This implies that

$$x_n - x_{n+1} = S^n x_0 - S^{n+1} x_0 \to 0$$

as $n \to \infty$. Thus S is asymptotically regular at x_0. Suppose that $r > 0$. Since

$$\|Sx_n - p\| = \|x_{n+1} - p\| \leq \|x_n - p\| \to r$$

as $n \to \infty$, it follows from the uniform convexity of X that

$$\|x_n - p - (Sx_n - p)\| \to 0$$

as $n \to \infty$. This implies

$$\|x_n - Sx_n\| = \|S^n x_0 - S^{n+1} x_0\| \to 0$$

as $n \to \infty$. Thus our proof is complete.

Corollary 3. *Let K be a subset of a uniformly convex Banach space X. Suppose that T is a selfmapping of K such that $F(T) \neq \emptyset$ and*

$$\begin{aligned}\|Tx - Ty\| &< a_1(x,y)\|x-y\| + a_2(x,y)(\|x-Tx\| + \|y-Ty\|) \\ &+ a_3(x,y)(\|x-Ty\| + \|y-Tx\|) \\ &+ a_4(x,y)\|x-y\|^{-1}\|x-Tx\|\,\|y-Tx\| \\ &+ a_5(x,y)\|x-y\|^{-1}\|x-Ty\|\,\|y-Tx\|\end{aligned}$$

for all x, y in K, $x \neq y$, where the mappings $a_i: X \times X \to [0,1]$ have the property

$$M = \sup_{x,y \in X} \{a_1(x,y) + 2a_2(x,y) + 2a_3(x,y) + a_4(x,y) + a_5(x,y)\} \leq 1$$

If there are $x_0 \in K$ and $b_i \in R_+$ for $i = 0, 1, \ldots, m$, with $\sum_{i=0}^{m} b_i = 1$ such that $S^n x_0$ is defined and lies in K for each $n \in N$, where $S: K \to K$ is defined by $S = \sum_{i=0}^{m} b_i T^i$, then $\|S^n x_0 - S^{n+1} x_0\| \to 0$ as $n \to \infty$, i.e., S is asymptotically regular at x_0.

Remark 7. *Taking* $a_1(x,y)=1$, $b_0 \in (0,1)$ *and* $b_2=\ldots = b_m = 0$ *in the Corollary 3, we obtain a result of Petryshyn and Williamson* [6] *(p. 457).*

Remark 8. *Taking* $a_i(x,y) = c_i \in R_+$ *for* $i=1,2,3$ *and* $a_4=a_5=0$ *in our Corolarry 3, we obtain a result of Ray and Chatterjee* [7, *Theorem 3.1*].

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