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THE TORSION TENSOR ON MANIFOLDS WITH r - TANGENT STRUCTURE

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1. The operators C and M in an r-tangent structure

As in the theory of almost tangent structures [1] M.P. GLOSS [8] introduce, in the theory of manifolds with r-tangent structure, operators C and M.

Let $V_{q(r+1)}$, $q=2s$, $r+1=2t$ be such manifold an let us denote be $\wedge^{c}_{r_1}(V_{q(r+1)})$ the vector space of all the complex-valued exterior r_1-formes defined on $V_{q(r+1)}$. We associate with r-tangent structure two operators C and M defined on $(V_{q(r+i)})$ in the following way:
If $(v_1,v_2,\ldots,v_{r_1})$ are r vectors of T^{c}_{x} and f an r_1-forme, we denote by $f(v_1,\ldots,v_{r_1})$ the value of f for $v_1\wedge v_2\wedge\ldots\wedge v_{r_1}$. C is then defined by the relation

$$Cf(v_1,\ldots,v_{r_1}) = f(Jv_1,Jv_2,\ldots,Jv_{r_1}) \qquad (1.1)$$

If the components of f are $f_{j_1j_2\ldots j_{r_1}}$ the components of Cf will be

$$(Cf)_{i_1i_2\ldots i_{r_1}} = F^{j_1}_{i_1}F^{j_2}_{i_2}\ldots F^{j}_{i_{r_1}}f_{j_1j_2\ldots j_{r_1}} \qquad (1.2)$$

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It is obvious that C satisfies the relation

$$C^{r+1} = 0 \tag{1.3}$$

The operator M will be defined in the following way:
For any $v_1, v_2, \ldots, v_{r_1} \in T^c_x$, M is defined by the relation

$$Mf(v_1, v_2, \ldots, v_{r_1}) = \sum_{k=1}^{r_1} f(v_1, v_2, \ldots, v_{k-1}, Jv_k, v_{k+1}, \ldots, v_{r_1}) \tag{1.4}$$

If the components of f are $f_{j_1 j_2 \ldots j_{r_1}}$ the components of Mf will be

$$(Mf)_{i_1 i_2 \ldots i_{r_1}} = \sum_{k=1}^{r_1} F^h_{i_k} f_{i_1 \ldots i_{k-1} h i_{k+1} \ldots i_{r_1}} \tag{1.5}$$

with respect to an r-adapted basis [4] :

$$F^j_i = \delta^q_{j-1} \tag{1.6}$$

Hence

$$(Cf)_{i_1 i_2 \ldots i_{r_1}} = \sum_{j_1, \ldots, j_{r_1}} \delta^q_{j_1 - i_1} \delta^q_{j_2 - i_2} \cdots \delta^q_{j_{r_1} - i_{r_1}} f_{j_1 j_2 \ldots j_{r_1}} =$$

$$= f_{i_1+q, i_2+q, \ldots, i_{r_1}+q} \tag{1.7}$$

and

$$(Mf)_{i_1 i_2 \ldots i_{r_1}} = \sum_{k=1}^{r_1} \delta^q_{h - i_k} f_{i_1 i_2 \ldots i_{k-1} h i_{k+1} \ldots i_{r_1}} =$$

$$= \sum_{k=1}^{r_1} f_{i_1 \ldots i_{k-1}, i_k+q, i_{k+1} \ldots i_{r_1}} \tag{1.8}$$

It is obvious that CM = MC.

Proposition 1. *For any 1-form f we have*

$$Cdf + dCMf - Mdcf = f \circ T \qquad (1.9)$$

where T is the tensor of the structure of the manifold $V_{q(r+1)}$.

Proof. We consider an 1-form f defined in a neighbourhood U of $V_{q(r+1)}$ by

$$f = f_i\, \theta^i.$$

Its exterior differential is

$$df = df_i \wedge \theta^i + f_i d\theta^i$$

$$df = \frac{1}{2}(\partial_i f_j - \partial_j f_i)\, \theta^i \wedge \theta^j + \frac{1}{2} f_i c^i_{jk} \theta^j \wedge \theta^k$$

Where c^i_{jk} are the coefficients of $d\theta^i$ in the decomposition

$$d\theta^i = \frac{1}{2} c^i_{jk}\, \theta^j \wedge \theta^k \quad (c^i_{jk} + c^i_{jk} = 0)$$

Operating by C, we find

$$Cdf = \frac{1}{2}(\partial_{i+q} f_{j+q} - \partial_{j+q} f_{i+q})\, \theta^i \wedge \theta^j + \frac{1}{2} f_i c^i_{j+q,k+q} \theta^i \wedge \theta^k$$

Similarly

$$Cf = f_{i+q} \theta^i$$

$$CMf = f_{i+q} \theta^i$$

$$CMf = f_{i+2q} \theta^i$$

$$dCMf = df_{i+2q} \wedge \theta^i + f_{i+2q} d\theta^i =$$

$$= \frac{1}{2} (\partial_l f_{i+2q} - \partial_{i+2q} f_l)\theta^l \wedge \theta^i + \frac{1}{2} f_{i+2q} C^i_{jk} \theta^j \wedge \theta^k$$

$$dCf = df_{i+q} \wedge \theta^i + f_{i+q} d\theta^i =$$

$$= \frac{1}{2} (\partial_l f_{i+q} - \partial_{i+q} f_l)\theta^l \wedge \theta^i + \frac{1}{2} f_{i+q} C^i_{jk} \theta^j \wedge \theta^k$$

$$MdCf = \frac{1}{2} (\partial_{l+q} f_{i+q} - \partial_{i+q} f_{l+q})\theta^l \wedge \theta^i + \frac{1}{2}(\partial_l f_{i+2q} - \partial_{i+2q} f_l)\theta^l \wedge \theta^i +$$

$$+ \frac{1}{2} f_{i+q} C^i_{j+q,k} \theta^j \wedge \theta^k + \frac{1}{2} f_{i+q} C^i_{j,k+q} \theta^j \wedge \theta^k$$

Therefore

$$Cdf + dCMf - MdCf = \frac{1}{2} (\partial_{i+q} f_{j+q} - \partial_{j+q} f_{i+q})\theta^i \wedge \theta^j +$$

$$+ \frac{1}{2} f_i C^i_{j+q,k+q} \theta^j \wedge \theta^k +$$

$$+ \frac{1}{2} (\partial_l f_{i+2q} - \partial_{i+2q} f_l)\theta^l \wedge \theta^i +$$

$$+ \frac{1}{2} f_{i+2q} C^i_{jk} \theta^j \wedge \theta^k -$$

$$- \frac{1}{2} (\partial_{l+q} f_{i+q} - \partial_{i+q} f_{l+q}) \theta^l \wedge \theta^i -$$

$$- \frac{1}{2} (\partial_l f_{i+2q} - \partial_{i+2q} f_l) \theta^l \wedge \theta^i -$$

$$- \frac{1}{2} f_{i+q} C^i_{j+q,k} \theta^j \wedge \theta^k -$$

$$- \frac{1}{2} f_{i+q} C^i_{j,k+q} \theta^j \wedge \theta^k =$$

$$\frac{1}{2} f_i [C^i_{j+q,k+q} + C^{i-2q}_{jk} - C^{i-q}_{j+q,k} - C^{i-q}_{j,k+q}]\theta^j \wedge \theta^k$$

Let us put

$$t^i_{jk} = C^i_{j+q,k+q} + C^{i-2q}_{jk} - C^{i-q}_{j+q,k} - C^{i-q}_{j,k+q}$$

the form (T^i) is then given by

$$T^i = \frac{1}{2}\, t^i_{jk}\, \theta^j \wedge \theta^k$$

and finally

$$Cdf + dCMf - MdCf = f_i T^i = f_o T.$$

2. Calculation of the tensor T.

The relation (1.9) can be used in order to obtain, in local coordinates, an expression of the tensor T in termes of the tensor F^j_i of the r-tangent structure.

Indeed (1.9) provides, in local coordinates

$$(Cdf + dCMf - MdCf)_{ij} = \frac{1}{2}\, t^i_{jk} f_i \qquad (2.1)$$

The form df has, as components in local coordinates

$$(df)_{jk} = \frac{1}{2}\,(\partial_j f_k - \partial_k f_j) \qquad (2.2)$$

$$(Cdf)_{jk} = \frac{1}{2}\, F^l_j F^m_k (\partial_l \varphi_m - \partial_m \varphi_l) \qquad (2.3)$$

$$(dCf)_{jk} = \frac{1}{2}\,[\partial_j (F^m_k f_m) - \partial_k (F^m_j f_m)] \qquad (2.4)$$

$$(MdCf)_{jk} = \frac{1}{2}\,\{F^l_j\,[\partial_l (F^m_k f_m) - \partial_k (F^m_l f_m)] - F^l_k [\partial_l (F^m_j f_m) - \partial_j (F^m_l f_m)]\}$$

$$= \frac{1}{2}\,\{F^l_j F^m_k \partial_l f_m - F^l_k F^m_j \partial_l f_m - [F^l_j F^m_l \partial_k f_m - F^l_k F^m_l \partial_j f_m] +$$

$$+ [F^l_j (\partial_l F^m_k - \partial_k F^m_l) - F^l_k (\partial_l F^m_j - \partial_j F^m_l)] f_m\} \qquad (2.5)$$

$$(dCMf)_{jk} = \frac{1}{2}\{\partial_j(F_k^l F_l^m f_m) - \partial_k(F_j^l F_l^m f_m)\} =$$

$$= \frac{1}{2}\{F_k^l F_l^m \partial_j f_m + F_k^l \partial_j F_l^m + \partial_j F_l^m f_m -$$

$$- F_j^l F_l^m \partial_k f_m - F_j^l \partial_k F_l^m f_m - \partial_k F_j^l F_l^m f_m\} \quad (2.6)$$

Therefore

$$(Cdf+dCMf-MdCf)_{jk} = \frac{1}{2}[F_l^m(\partial_j F_k^l - \partial_k F_j^l) - F_j^l \partial_j F_k^m + F_k^l \partial_l F_j^m] f_m \quad (2.7)$$

From (2.1) we see that

$$t_{jk}^i = F_l^i(\partial_j F_k^l - \partial_k F_j^l) + F_k^l \partial_l F_j^i - F_j^l \partial_l F_k^i \quad (2.8)$$

Proposition 2. *For the r-tangent structures the Nijenhuis tensor, is the negative of the tensor of the structure.*

Proof. The Nijenhuis tensor is defined [5] by

$$N(u,v) = [Ju,Jv] + J^2[u,v] - J[Ju,v] - J[u,Jv] \quad (2.9)$$

for any vector fields u,v.
The relation (2.9) may be written explicitly

$$[N(u,v)]^k = [Ju,Jv]^k + F_s^k F_l^s [u,v]^l - F_l^k [Ju,v]^l - F_l^k [u,Jv]^l =$$

$$= F_a^m u^a \partial_m (F_b^k v^b) - F_a^m v^a \partial_m (F_b^k u^b) +$$

$$+ F_s^k F_l^s [u^m \partial_m v^l - v^m \partial_m u^l] - F_l^k [F_a^m u^a \partial_m v^l - v^m \partial_m (F_b^l u^b)] -$$

$$- F_l^k [u^m \partial_m (F_a^l u^a) - F_b^m v^b \partial_m u^l] =$$

$$= F_a^m F_b^k u^a \partial_m v^b - F_a^m F_b^k v^a \partial_m u^b + F_a^m \partial_m F_b^k u^a v^b - F_a^m \partial_m F_b^k v^a u^b +$$

$$+ F_s^k F_l^s u^m \partial_m v^l - F_s^k F_l^s v^m \partial_m u^l - F_l^k F_a^m u^a \partial_m v^l +$$

$$+ F_l^k v^m [F_b^l \partial_m u^b + u^b \partial_m F_b^l] - F_l^k u^m [F_b^l \partial_m v^b + v^b \partial_m F_b^l] +$$

$$+ F_l^k F_b^m v^b \partial_m u^l = [F_l^k (\partial_b F_a^l - \partial_a F_b^l) + F_a^m \partial_m F_b^k - F_b^m \partial_m F_a^k] u^a v^b \partial$$

Therefore

$$N_{jk} = F_l^i (\partial_k F_j^l - \partial_j F_k^l) + F_j^l \partial_l F_k^i - F_k^l \partial_l F_j^i$$

and finally

$$N_{jk}^i = -t_{jk}^i$$

Corollary 1. *For any 1-form f we have*

$$Cdf + dCMf - MdCf = -f \circ N$$

Corollary 2. *In order that an r-tangent structure be completely integrable it is necessary and sufficient that the Nijenhuis tensor be equal to zero.*

Corollary 3. *In order that an r-tangent structure be completely integrable it is necessary and sufficient that for any 1-form f we have*

$$Cdf + dCMf - MdCf = 0.$$

References.

[1] H.A. ELIOPOULOS. On the general theory of differentiable manifolds with almost tangent structure Canad.Math. Bull. Vol.8, No 6, 1965, pp. 721-748.

[2] H.A. ELIOPOULOS. Varietés r-tangentes, Comptes rendus , 263 (1966) pp. 413-416.

[3] H.A. ELIOPOULOS. Some properties of differentiable structures defined by niplotent operators of degree r. Bull. Soc. Math. Gréce, 7 (1966), pp. 81-94.

[4] H.A. ELIOPOULOS. Hermitian structures subordinated to r-tangent structures. Tensor, N.S. Vol. 25 (1972) pp. 319-320.

[5] A. FLOLICHER AND A. NIJENHUIS. Theory of vector - valued differential formes, part I, K. Ned. Akad. Wet Proc. Math. Sc. V59, pp. 338-359. 1956.

[6] M^{me} J. LEHMANN-LEJEUNE. Sur l'integrabilité de certain G-structures, C.R. Acad. Sc. Paris, t258 pp. 5326-5329, 1964.

[7] A. LICHNEROWICZ. Théorie grobale des connexions et des groupes d'holonomie. Edizioni Gremonese, Roma 1962.

[8] M. GLOSS. Les operateurs C et M sur les varietes r-tan - gent C.R. Acad. Sc. Paris, t 267, p.28-31, 1968.

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