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A NON-STANDARD CHARACTERIZATION OF THE NORM OF FREE ULTRAFILTERS

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An important tool for the non-standard treatment of Analysis and Topology is the notion of filter monad, defined and investigated by Luxemburg in [3]. Filter monad is the non-standard equivalent of filter. So, in principle, any property of a filter, can be translated into a property of its monad and vice-versa. In practice, however, this translation is by no means easy and only a few kinds of filters have been so far characterized by non-standard conditions.

In this note we define a notion of ordinal ranking for the monads of ultrafilters (ultramonads) and we prove the following:

Theorem. *For any ultrafilter F on a set X, the norm of F is $\aleph_\beta$ iff the rank of its monad is β.*

We assume familiarity of the reader with the basic notions of Non-Standard Analysis and especially Robinson's

Theory of Enlargements, as well as the set-theoretic properties of monads. For details one should confere, for example, [2], [3], [4] or [6].

Remarks on notation and terminology. *If X is a set of a given structure M and we take an arbitrary but fixed enlargement *M of M, then *X denotes the corresponding set of *M which always includes X. If X is infinite, then *X strictly includes X and *X-X is infinite. Let F be a (proper) filter on X. The monad of F (with respect to the enlargement *M) is the set*

$$\mu(F) = \cap\{{}^*V \mid V \in F\}$$

A filter F is free if $\cap\{V \mid V \in F\} = \emptyset$. *Norm of a filter F is the cardinal number.*

$$\|F\| = \min\{|V| \mid V \in F\}$$

where | | denotes cardinality. We write $\alpha, \beta, \gamma, \ldots$ *for ordinal numbers and* $\aleph_\alpha, \aleph_\beta, \aleph_\gamma, \ldots$ *for cardinal numbers.*

C. Puritz ([5]) calls an element $x \in {}^*X$ *serial if there is some countable* $A \subset X$ *such that* $x \in {}^*A$. *The definition is non-trivial if X is uncountable. In such a case it is proved that all non-serial elements of *X constitute the monad of the filter generated by the complements of all countable subsets of X.*

*We extend, here, this classification of the elements of *X in a natural way.*

Let $|X| = \aleph_\alpha$. *For each* $\beta \leq \alpha$, *define on X the filter*

$$F_\beta = \{V \subset X \mid |X - V| < \aleph_\beta\}$$

(For these filters see also [1]). It is easy to check that all F_β *are free. This means that their monads do not contain elements of X, i.e. they are subsets of *X-X. Some other*

properties of F_β are given by the following:

Lemma 1. *(i)* $\beta < \gamma \leq \alpha \rightarrow F_\beta \subsetneq F_\gamma$

(ii) For any limit ordinal $\beta \leq \alpha$, $F_\beta = \sup\{F_\gamma | \gamma < \beta\}$

(iii) None of F_β *is an ultrafilter.*

Proof. (i) $\beta < \gamma \rightarrow \aleph_\beta < \aleph_\gamma$. Let $V \in F_\beta$. Then $|X-V| < \aleph_\beta < \aleph_\gamma$ hence $V \in F_\gamma$. Now take $V \in F_\gamma$ such that $|X-V| = \aleph_\beta$. Cleary $V \in F_\beta$.

(ii) In vew of (i) evidently $\sup\{F_\gamma | \gamma < \beta\} \subset F_\beta$.

Let $V \in F_\beta$. Then $|X-V| < \aleph_\beta$ and suppose $|X-V| = \aleph_\gamma$ for some $\gamma < \beta$. It follows that $V \in F_\gamma$ and since $F_\gamma \subset \sup\{F_\gamma | \gamma < \beta\}$, finally $F_\beta = \sup\{F_\gamma | \gamma < \beta\}$.

(iii) It suffices to show that F_α is not an ultrafilter. Take a partition of X into two sets X_1, X_2, each of power $\aleph_\alpha$ It is evident that neither X_1 nor X_2 belong to F_α and thus F_α is not an ultrafilter.

We now set $S_\beta = \mu(F_\beta)$. In virtue of (i) above

$$\beta < \gamma \leq \alpha \rightarrow S_\gamma \subsetneq S_\beta \qquad \text{(i)}$$

where $S_0 = {}^*X - X$.

Definition 2. An element $x \in {}^*X - X$ is said to be of rank $\beta, \beta, <\alpha$- in symbols $r(x) = \beta$- if $x \in S_\beta - S_{\beta+1}$ and $r(x) = \alpha$ if $x \in S_\alpha$. The elements of X are, by definition, of rank zero.

In this terminology, the serial elements of Puritz are, exactly, those of rank zero.

Lemma 3. *To each element of* ${}^*X$ *there is associated one and only one rank.*

Proof. We have to show that the sets $S_\beta - S_{\beta+1}$, $\beta \neq \alpha$ and S_α form a partition of ${}^*X - X$, i.e. they are disjoint and their union equals ${}^*X - X$. The first assertion is obvious. As for the second we observe that

$$\bigcup_{\beta<\alpha} (S_\beta - S_{\beta+1}) \cup S_\alpha = (\bigcap_{\beta<\alpha} S_{\beta+1})' \cup S_\alpha$$

where ' means complementation w.r.t. *X-X.

If α is a succesor ordinal, then S_α is the least monad in the nested sequence (i)' and so

$$\bigcap_{\beta<\alpha} S_{\beta+1} = S_\alpha .$$

If α is a limit ordinal, it follows from L.2 (ii) and the properties of monads that again

$$\bigcap_{\beta<\alpha} S_{\beta+1} = \bigcap_{\beta<\alpha} S_\beta = S_\alpha ,$$

and the lemma is proved.

The following facts are known from the literature([4]). For any $x \in {}^*X$ the set

$$F(x) = \{V \subset X \mid x \in {}^*V\}$$

is an ultrafilter on X and F(x) is trivial, i.e. non-free (resp. free), iff $x \in X$ (resp. $x \in {}^*X - X$). Conversely any ultrafilter on X can be written in the form F(x), where x is any element of its monad. Finally, any monad is a union of ultramonads, and any two ultramonads are disjoint.

So, if $\mu(F)$ is an ultramonad, $x \in \mu(F)$ and $r(x) = \beta$ ($x \in S_\beta - S_{\beta+1}$), it is clear that all elements of $\mu(F)$ will be of rank β ($\mu(F) \subset S_\beta - S_{\beta+1}$). So we can extend Def.2 to the following.

Definition 4. The rank of an ultramond $\mu(F)$ is if it contains elements of rank .

Proof of the Theorem. We shall prove that for any free ultrafilter F

$$\|F\| = \aleph_\beta \leftrightarrow r(\mu(F)) = \beta$$

Sufficiency. Let $r(\mu(F)) = \beta$ and $x \in \mu(F)$. Then

$F = F(x)$ and $x \in S_\beta - S_{\beta+1}$. Now $x \in S_\beta$ implies $|V| \geq \aleph_\beta$ for all $V \in F$. For if $V \in F$ and $|V| < \aleph_\beta$, then $X - V \in F_\beta$ whence $S_\beta \subset {}^*X - {}^*V$. It means $x \notin {}^*V$ and this contradicts the fact that $V \in F = F(x)$ i.e. $x \in {}^*V$. Hence $|V| \geq \aleph_\beta$ for all $V \in F$, so $\|F\| \geq \aleph_\beta$. On the other hand $x \notin S_{\beta+1}$ implies that there is some $V \in F_{\beta+1}$ such that $x \notin {}^*V$ or $x \in {}^*X - {}^*V$ with $|X - V| \leq \aleph_\beta$. Since $X - V \in F$ it follows that $\|F\| \leq \aleph_\beta$.

Necessity. Let $\|F\| = \aleph_\beta$, and $V \in F_\beta$. Then $|X - V| < \aleph_\beta$ and obviously $X - V$ cannot belong to F, so $V \in F$. Thus $F_\beta \subset F$ which means that $x \in S_\beta$. Now there is some $V \in F$ such that $|V| = \aleph_\beta$. Since $X - V \in F_{\beta+1}$ and $x \notin {}^*X - {}^*V$ it follows that $x \in S_{\beta+1}$.

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