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UNIFORMLY REGULAR SETS OF MEASURES ON COMPLETELY REGULAR SPACES

By
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1. Introduction and preliminary matrial.

The concept of uniform regularity of measures was motivated by the following simple observation: If μ is a Baire meausre on a metric space X with metric d, and Q is a closed subset of X then, writing $Q_n = \{x \in X: d(x,Q) < \frac{1}{n}\}$ the σ-additivity of μ gives: $\mu(Q) = \lim_n \mu(Q_n)$. The use of a sequence of elements of an admissible uniformity, instead of the metric d, lead to the introduction of the concept of uniform regularity for measures on compact spaces in [3] (extended to measures on arbitrary completely regular spaces in [4]). Such measures behave in many ways like measures on metric spaces e.g. they have separable supports, separable L^p-spaces ($1 \leq p < \infty$), [3], and their associated topological measure spaces admit some form of a representation by metric measure spaces [2].

In this note we study sets of measures for which a single sequence of the uniformity can be used to approximate each measure in the set on zero subsets of the underlying topological space. Sets of measures enjoing this form of joint uniform regularity are called uniformly regular sets. Structural properties of uniformly regular sets of signed measures are discussed in §2 where it is shown that the linear hull and the norm closure of a uniformly regular set is uniformly regular. §3 deals with uniform regularity of sets of measures relative to the iniformity $\mathcal{G}^*$ generated by all the bounded continuous functions on the space X. We show that a $\mathcal{G}^*$-uniformly regular set of measures is characterized by the property that the corresponding functionals are uniquely determined by their values on a fixed norm separable subalgebra of $C^*(X)$. This is used to show that, relative to the weak topology on $\mathcal{M}(X)$ the space of all measures on X, where X is pseudo-compact, the compact G_δ-metrizable subsets of $\mathcal{M}(X)$ are precisely the compact $\mathcal{G}^*$-uniformly regular sets. As a consequence of this we obtain a strengthened generalized version of a result of [1] where a topological characterization of uniformly regular probability measures on compact spaces was given.

Throughout, X will denote a completely regular Hausdorff space, and $C^*(X)$ the Banach algebra of all bounded realvalued continuous functions on X under the pointwise algebraic operations and the supremum norm. A(Baire) measure is a σ-additive non-negative finite set function defined on the Baire sets. A signed measure is a difference of two measures. When it is convenient, we shall identify signed measures on X with the σ-additive linear functionals on $C^*(X)$ regarded as a Banach space [8]. We denote by $\mathcal{M}(X)$ and $\mathcal{M}^+(X)$ (or just $\mathcal{M}$ and $\mathcal{M}^+$ when no confusion car arise) the linear space of all signed measures on X and the space of all measures on X respectively. The strong (or the norm) topology on $\mathcal{M}$ is the topology

induced by the norm $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the total variation of μ; and the weak topology is the topology induced by the duality $< \mathcal{M}^+, C^*(X) >$.

Let $\mathcal{U}$ be an admissible uniformity of X (i.e. a uniformity inducing the given topology on X) and μ a measure on X. Following [3] and [4], we say that μ is uniformly regular relative to $\mathcal{U}$ (or just $\mathcal{U}$-uniformly regular) if there exists a sequence $\{U_n\}$ of elements of $\mathcal{U}$ such that for any zero set $Z \subset X$, $U_n(Z)$ is a Baire set for all n and $\mu(Z)=\lim_n \mu(U_n(Z))$. $\mathcal{U}$-uniform regularity of a signed measure μ is defined by the requirement that $|\mu|$ is $\mathcal{U}$-uniformly regular. When X is compact, no mention of a uniformity is necessary since there is a unique admissible uniformity of X. In this case a uniformly regular measure μ has a separable support and has the property that the Baire σ-algebra is equivalent to a countably generated sub-σ-algebra [3]. Moreover, there is a compact metric space T and a continuous surjection $p:X \to T$ such that for any μ-measurable set E we have, $\mu(E)=p(\mu)(p(E))$ [2]. i.e. such measures look, in many ways, like measures on metric spaces.

All unexplained topological notions will be those of [7] and [5]. For the measure theoretic notions we follow the usage in [6].

2. Uniform regularity of sets of signed measures.

Let $\mathcal{U}$ be an admissible uniformity of X and $\mathcal{N} \subset \mathcal{M}(X)$ a set of signed measures on X. We say that $\mathcal{N}$ is uniformly regular relative to $\mathcal{U}$ (or just $\mathcal{U}$-uniformly regular if there exists a decreasing sequence $\{U_n\}$ of elements of such that for any $\mu \in \mathcal{N}$ and any zero set $Z \subset X$ we have, $U_n(Z) = \{x \in X: (x,y) \in U_n \text{ for some } y \in Z\}$ is a Baire set, for all n, and

$$|\mu|(Z) = \lim_n |\mu|(U_n(Z))$$

We note that the word "uniform" in the above definition is used in the sense that a single sequence $\{U_n\}$ works for all zero sets Z and is not meant to suggest that $|\mu|(U_n(Z))$ aproximates $|\mu|(Z)$ uniformly in Z. We also observe that any uniformity $\mathfrak{U}$ has a subbase consisting of elements of the form

$$U = \{(x,y): d(x,y) < \varepsilon\}$$

where d is a continuous pseudometric and $\varepsilon>0$. Since any such U is clearly symmetric and is such that $U(Z) = \{x:d(x,Z) < \varepsilon\}$ is a cozero set, we may assume that each U_n in the above definition is symmetric.

When X is compact, $\mathfrak{U}$ is the unique admissible uniformity of X. In this case we refer to $\mathcal{N}$ simply as a uniformly regular set of signed measures.

It is clear from the definition that any countable union of $\mathfrak{U}$-uniformly regular sets of signed measures is $\mathfrak{U}$-uniformly regular. Moreover we have.

2.1. Proposition. *Let $\mathfrak{U}$ be an admissible uniformity of X and suppose that $\mathcal{N} \subset \mathcal{M}(X)$ is $\mathfrak{U}$-uniformly regular. Then the strong closure of the linear subspace generated by $\mathcal{N}$ is $\mathfrak{U}$-uniformly regular.*

Proof. Let $\{U_n\}$ be a decreasing sequence of elements of $\mathfrak{U}$ such that $|\mu|(Z) = \lim_n |\mu|(U_n(Z))$, for all $\mu \in \mathcal{N}$ and all zero sets $Z \subset X$. Denote by $\mathfrak{L}$ the set of all signed measures μ such that $|\mu|(Z) = \lim_n |\mu|(U_n(Z))$ for all zero sets $Z \subset X$. It is sufficient to show that $\mathfrak{L}$ is a strongly closed linear subspace of $\mathcal{M}(X)$. Suppose that $\{\mu_m\}$ is a sequence of signed measures in $\mathfrak{L}$ such that $\mu_m \to \mu$ strongly. Let $Z \subset X$ be a zero set and write $E = [\bigcap_n U_n(Z)] \setminus Z$. If $A \subset X$ and $B \subset X$ form a Hahn de-

composition of X with respect to μ, we have,

$$\mu^{+}(E) = \mu(E \cap A) = \lim_{m} \mu_m(E \cap A) \leq \lim_{m} |\mu_m|(E) = 0.$$

Similarly $\mu^{-}(E) = 0$, and so $|\mu|(E) = 0$. i.e. $\mu \in \mathfrak{L}$. Thus $\mathfrak{L}$ is strongly closed. It is easily seen that any μ in the linear hull of $\mathfrak{L}$ satisfies $|\mu|(E) = 0$ and so $\mathfrak{L}$ is a linear subspace of $\mathcal{M}(X)$.

If X is metrizable then $\mathcal{M}(X)$ is obviously uniformly regular relative to the metric uniformity. This property characterizes metric spaces.

2.2. Proposition. *X is metrizable if and only if $\mathcal{M}(X)$ is uniformly regular relative to some admisible uniformity of X.*

Proof. Suppose that $\{U_n\}$ is a sequence of elements of an admissible uniformity of X such that $|\mu|(Z) = \lim_{n} |\mu|(U_n(Z))$ for all $\mu \in \mathcal{M}(X)$ and all zero sets $Z \subset X$. We assume that U_n is symmetric for all n (see the remark following the definition of $\mathfrak{U}$-uniform regularity). Let $x \in G \subset X$, where G is a cozero subset of X. If μ is the Baire measure induced by a unit-point-mass at x then, since $\mu(X \setminus G) = 0$, there exists n such that $x \notin U_n(X \setminus G)$. It follows that $U_n(x) \subset G$. Thus the uniformity generated by the sequence $\{U_n\}$ is admissible, and hence X is metrizable. The converse is obvious.

Proposition (2.1) shows that the strong closure of a uniformly regular set of signed measures (relative to a given uniformity) is uniformly regular. We give an example of a uniformly regular set whose weak closure is not uniformly regular.

2.3. Example. *Let X be the (Helly) space of all non-decreasing functions from the closed unit interval into*

itself with the pointwise convergence topology. X is compact, Hausdorff, first countable, separable but not metrizable [7], ch. V. Let $\{x_n\}$ be dense sequence in X and for each n let μ_n be the Baire measure induced by a unit-point-mass at x_n. Since each point of X is a G_δ, the measure μ_n is uniformly regular [3]. It follows from proposition (2.1) that the linear space $\mathfrak{L}$ generated by $\{\mu_n; n=1,2,\ldots\}$ is uniformly regular. is weakly dense in $\mathcal{M}(X)$. It follows from proposition (2.2) and the fact that X is not metrizable that $\mathcal{M}(X)$, the weak closure of $\mathfrak{L}$ is not uniformly regular.

3. $\mathfrak{S}^*$-Uniform regularity.

The uniformity $\mathfrak{S}^*$ generated by all the bounded continuous functions on X is a natural uniformity to consider for completely regular spaces. In this section we shall be concerned with $\mathfrak{S}^*$-uniform regularity of subsets of $\mathcal{M}^+(X)$ the space of all Baire measures on X. For $f \in C^*(X)$ and $\mu \in \mathcal{M}^+(X)$ we write

$$\langle f,\mu\rangle = \int_X f d\mu$$

All topological statements about $\mathcal{M}^+(X)$ will be relative to the weakest topology making each of the maps $\mu \to \langle f,\mu\rangle$, $f \in C^*(X)$ continuous.

The following theorem is a $\mathfrak{S}^*$-version of proposition (2.2).

3.1. Theorem. *The following conditions on a completely regular Hausdorff space X are equivalent.*

(i) X is separable and metrizable.

(ii) $\mathcal{M}(X)$ is $\mathfrak{S}^*$-uniformly regular.

(iii) $\mathcal{M}^+(X)$ is $\mathfrak{S}^*$-uniformly regular.

Proof. Suppose that X is a separable metric space with a bounded metric d. Let $\{x_n\}$ be a dense sequence in X and for each n define $f_n(x)=d(x,x_n)$. f_n is continuous. For $x,y\in X$ let $\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x)-f_n(y)|$. Then ρ is an admissible metric on X generating a subuniformity of $\mathfrak{G}^*$. It follows that $\mathcal{M}(X)$ is $\mathfrak{G}^*$-uniformly regular. i.e. (i) ⇒ (iii)
(ii) ⇒ (iii) is obvious. We now show that (iii) ⇒ (i).

Suppose that $\mathcal{M}^+(X)$ is $\mathfrak{G}^*$-uniformly regular. The method used in proving proposition (2.2) shows that there is an admissible countably generated subuniformity $\mathfrak{U}$ of $\mathfrak{G}^*$. The completion of X relative to $\mathfrak{U}$ is a compact metric space and hence X is second countable and therefore separable and metrizable.

We shall make use of the following functional characterization of $\mathfrak{G}^*$-uniformly regular sets of measures. The "only if" part of the proof is essentially the same as that given in theorem (4.2) of [4] where the result is given for a single measure.

3.1. Theorem. *Let $\mathcal{N}\subset\mathcal{M}^+(X)$. Then $\mathcal{N}$ is $\mathfrak{G}^*$-uniformly regular if and only if there is a norm separable subalgebra B of $C^*(X)$ such that for any $\mu\in\mathcal{N}$ and any $f\in C^*(X)$ with $f\geq 0$, we have,*

$$\langle f,\mu\rangle = \inf\{\langle g,\mu\rangle : g\in B,\ g\geq f\}$$

Proof. Suppose that $\mathcal{N}$ is $\mathfrak{G}^*$-uniformly regular, and let $\{U_n\}$ be a sequence of elements of $\mathfrak{G}^*$ approximating each $\mu\in\mathcal{N}$ on zero sets. We may assume that U_n has the form

$$U_n = \{(x,y): |f_i(x)-f_i(y)| \leq \varepsilon_n\ ,\ i = 1,2,\ldots,n\}$$

where $\{f_i\}$ is a sequence of functions in $C^*(X)$ and $\{\varepsilon_n\}$ is a sequence of positive numbers decreasing to 0. Let B be the smallest closed subalgebra of $C^*(X)$ containing each f_i and the constants. B is norm separable. Thus there is a compact Hausdorff space T such that B is isometrically isomorphic to C(T), the algebra of all realvalued continuous functions on T. For each $g \in B$, denote by $\tilde{g}$ the image of g under this isomorphism. This isomorphism induces a continuous map $p:X \to T$ where p(x) is the unique point in T such that $\tilde{g}(p(x))=0$ for all $g \in B$ satisfying $g(x) = 0$.

Now, let $\mu \in \mathcal{N}$ and $Z \subset X$ be a zero set.

Writing

$$U_n = \{(s,t) \in T*T: |\tilde{f}_i(s)\tilde{f}_i(t)| \leq \varepsilon_n, \ i = 1,2,\ldots,n\}$$

and defining the measure λ on T by:

$$\langle g,\lambda\rangle = \langle g,\mu\rangle \ , \text{ for all } \tilde{g} \in C(T)$$

and denoting by I_E the indicator function of a set E, we have, $\tilde{U}^2_{n+1} = \tilde{U}_{n+1} \cdot \tilde{U}_{n+1} \subset U_n$ for all n, and

$$\mu(U_n(Z)) = \lambda(\tilde{U}_n(p(Z))) \geq \lambda(\tilde{U}_{n+1}(p(Z))) \geq \lambda[u_T(\tilde{U}_{n+1}(p(Z)))]$$

$$\geq \inf\{\langle \tilde{g},\lambda\rangle : \tilde{g} \in C(T), \tilde{g} \geq I_{p(U_n(Z))}\}$$

$$= \inf\{\langle g,\mu\rangle : g \in B, \ g \geq I_{U_n(Z)}\}$$

$$\geq \inf\{\langle g,\mu\rangle : g \in B, \ g \geq I_Z\}$$

$$\geq \mu(Z).$$

taking limit as $n \to \infty$ we have

$$\mu(Z) = \inf\{\langle g,\mu\rangle : g \in B, \ g \geq I_Z\}$$

This extend to

$$\langle f,\mu\rangle = \inf\{\langle g,\mu\rangle: g\in B,\ g \geq f\}$$

for all $f\in C^*(X)$, $f \geq 0$ since such f can be approximated from above in $L^1(\mu)$ by functions of the form $\sum_{j=1}^{n} \alpha_j I_{Z_j}$ where Z_j is a zero set and $\mu(Z_i \cap Z_j) = 0$, if $i \neq j$.

Conversely suppose that there is a norm separable sub-algebra B of $C^*(X)$ such that for any $\mu \in \mathcal{N}$ and any $f\in C^*(X)$ $f \geq 0$, we have

$$\langle f,\mu\rangle = \inf\{\langle g,\mu\rangle: g\in B,\ g \geq f\}$$

It follows that

$$\mu(Z) = \inf\{\langle g,\mu\rangle: g\in B,\ g \geq I_Z\},$$

for all $\mu \in \mathcal{N}$ and all zero sets $Z\subset X$.

Let $\{g_n\}$ be a dense sequence in B and write

$$U_n = \{(x,y): |g_i(x)-g_i(y)| \leq 2^{-n},\ i = 1,2,\ldots,n\}$$

For any $\mu \in \mathcal{N}$, Z a zero subset of X, let $g\in B$ be such that $g \geq I_Z$. We show that $g \geq I_{\tilde{Z}}$ where $\tilde{Z} = \bigcap_n U_n(Z)$.

Let $y\in \tilde{Z}$. Then for each n there exists $x_n \in Z$ such that $|g_n(y)-g_n(x_n)| < 2^{-n}$.

For any $\delta>0$ let m be an integer such that $\delta>2^{-m+1}$ and $\|g-g_m\|<\delta/4$. Hence

$$g(y) > g_m(y)-\delta/4 > g_m(x_m)-2^{-m}-\delta/4 > g_m(x_m)-3\delta/4$$
$$> g(x_m)-\delta \geq 1-\delta.$$

Therefore $g(y) \geq 1$ and thus $g \geq I_{\tilde{Z}}$.

It follows that $\mu(Z) = \mu(\tilde{Z}) = \lim_n \mu(U_n(Z))$
i.e. $\mathcal{N}$ is *-uniformly regular. This completes the proof.

We now give a topological characterization of certain $\mathfrak{G}^*$-uniformly regular sets of measures.

3.2. Theorem. *Suppose that X is pseudocompact and $\mathcal{N}$ is a closed uniformly bounded set of measures on X. Then $\mathcal{N}$ is $\mathfrak{G}^*$-uniformly regular if and only if $\mathcal{N}$ is a metrizable G_δ-subset of $\mathcal{M}^+(X)$.*

Proof. Suppose that $\mathcal{N}$ is $\mathfrak{G}^*$-uniformly regular. It follows from theorem (3.2) that there is a norm separable subalgebra B of $C^*(X)$ containing the constant functions, such that for any $\mu \in \mathcal{N}$ and any $f \in C^*(X)$, $f \geq 0$, we have,

$$\langle f,\mu\rangle = \inf\{\langle g,\mu\rangle : g \in B,\ g \geq f\}$$

Let $\{g_n\}$ be a dense sequence in B and write

$$V_n = \{(\mu,\nu) \in \mathcal{M}^+ \times \mathcal{M}^+ : |\langle g_i,\mu\rangle - \langle g_i,\nu\rangle| < 2^{-n},\ i=1,2,\ldots,n\}$$

Pseudocompactness of X and the fact that $\mathcal{N}$ is uniformly bounded and weakly closed imply that $\mathcal{N}$ is compact in $\mathcal{M}^+$. A simple compactness argument can be used to show that

$$\bigcap_n V_n(\mathcal{N}) = \bigcup_{\mu \in \mathcal{N}} (\bigcap_n V_n(\mu))$$

So, if $\nu \in \bigcap_n V_n(\mathcal{N})$, there exists $\mu \in \mathcal{N}$ such that $\nu \in V_n(\mu)$ for all n. i.e. $\langle g_n,\nu\rangle = \langle g_n,\mu\rangle$ for all n and hence $\langle g,\nu\rangle = \langle g,\mu\rangle$ for all $g \in B$.

Thus, for any $f \in C^*(X)$, $f \geq 0$, we have,

$$\begin{aligned}\langle f,\mu\rangle &= \inf\{\langle g,\mu\rangle : g \in B,\ g \geq f\} \\ &= \inf\{\langle g,\nu\rangle : g \in B,\ g \geq f\} \\ &\geq \langle f,\nu\rangle\end{aligned}$$

So, $\nu(E) \leq \mu(E)$ for all Baire $E \subset X$.

But $\nu(X) = \langle 1,\nu\rangle = \langle 1,\mu\rangle = \mu(X)$.

It follows that $\mu=\nu$ and hence $\mathcal{N}$ is a G_δ in $\mathcal{M}^+$. Moreover, it is clear that

$$\bigcap_n (V_n \cap (\mathcal{N} \times \mathcal{N})) = \{(\mu,\mu): \mu \in \mathcal{N}\}$$

So that the diagonal of $\mathcal{N} \times \mathcal{N}$ is a G_δ. Compactness of $\mathcal{N}$ implies that $\mathcal{N}$ is metrizable.

Conversely suppose that $\mathcal{N}$ is a G_δ metrizable subset of $\mathcal{M}^+$. This assumption together with the compactness of $\mathcal{N}$ give two sequences $\{V_n\}$ and $\{V'_n\}$ of the uniformity of $\mathcal{M}^+$ generated by functions in $C^*(X)$, such that

$$\bigcap_n V_n(\mathcal{N}) = \mathcal{N} \quad \text{and,}$$

$$\bigcap_n (V'_n \quad (\mathcal{N} \times \mathcal{N})) = \{(\mu,\mu): \mu \in \mathcal{N}\}$$

and so a sequence $\{W_n\}$ of this uniformity can be found such that for all $\mu \in \mathcal{N}$ we have,

$$\bigcap_n W_n(\mu) = \{\mu\} \quad (*)$$

We may assume that W_n has the form:

$$W_n = \{(\mu,\nu) \in \mathcal{M}^+ \times \mathcal{M}^+ : |\langle g_i,\mu\rangle - \langle g_i,\nu\rangle| < \varepsilon_n,\ i = 1,2,\ldots,n\}$$

where $\{g_n\}$ is a sequence of functions in $C^*(X)$ and $\{\varepsilon_n\}$ is a sequence of positive numbers decreasing to zero.

Let B be the closed subalgebra of $C^*(X)$ generated by the sequence $\{g_n\}$ and the constant functions. It then follows from (*) that for any $\mu \in \mathcal{N}$ and any $\nu \in \mathcal{M}^+$ we have

$$\langle g,\nu\rangle = \langle g,\mu\rangle \text{ for all } g \in B \Rightarrow \mu = \nu$$

We show that for any $\mu \in \mathcal{M}$ and any $f \in C^*(X)$, $f \geq 0$ we have

$$\langle f,\mu\rangle = \inf\{\langle g,\mu\rangle: g \in B,\ g \geq f\}$$

Suppose, to the contrary, that there is a measure $\mu \in \mathcal{N}$ and a function $f \in C^*(X)$, $f \geq 0$ such that

$$\langle f,\mu\rangle < \alpha = \inf\{\langle g,\mu\rangle: g \in B,\ g \geq f\}$$

Define the linear functional L on $H = B \oplus Rf$ the linear subspace of $C^*(X)$ generated by B and f, by

$$L(g+cf) = \langle g,\mu\rangle + c\alpha \qquad (g \in B)$$

L is positive since if $g+cf \geq 0$ and $c>0$ then

$$L(g+cf) = \langle g,\mu\rangle + c\alpha \geq \langle g,\mu\rangle + c\langle f,\mu\rangle \geq 0;$$

and if $g+cf \geq 0$ and $c<0$ then $-\frac{g}{c} \geq f$ so that $\langle -\frac{g}{c}, \mu\rangle \geq \alpha$. i.e. $0 \leq \langle g,\mu\rangle + c\alpha = L(g+cf)$. Let $\tilde{L}$ be any positive extension of L to $C^*(X)$. Since X is pseudocompact $\tilde{L}$ is σ-additive and there is a Baire measure λ s.t. $L(f) = \langle f,\lambda\rangle$ for all $f \in C^*(X)$. But $\langle g,\lambda\rangle = \langle g,\mu\rangle$ for all $g \in B$. Therefore $\lambda = \mu$- a contradiction. i.e. $\langle f,\mu\rangle = \inf\{\langle g,\mu\rangle: g \in B,\ g \geq f\}$, for all $f \in C^*(X)$, $f \geq 0$ and for all $\mu \in \mathcal{N}$. It follows form theorem (3.1) that $\mathcal{N}$ is $\mathfrak{G}^*$-uniformly regular. This completes the proof.

The following corollary which strengthens and extends th. (5.2) of [1], follows imediately from theorem (3.2).

3.3. Corollary. *A Baire measure μ on a pseudocompact space X is $\mathfrak{G}^*$-uniformly regular if and only if μ is a G_δ-point in $\mathcal{M}^+(X)$.*

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