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A NEW LINE INVOLUTION AND THE CLASSIFICATION PROCESS

By
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In this paper we undertake to introduce the classification of line involutions in a projective space of three dimensions, S_3, over the field of complex numbers. In doing so we present a classification table to be completed with the characteristics of known involutions and hopefull new ones to be discovered. The discovery and incorporation into the table of new line involutions we demonstrate by presenting a new involution called the twisted-cubic line involution.

Now, when we speak of an involution we mean a projective transformation of period two, that is, a transformation T such that T followed by T equals the identity transformation. The study of line configurations of a three dimensional projective space, S_3, is often greatly facilitated by considering these line configurations as point configurations in a projective space of higher dimension. Thus, we assume that the reader is familiar with Plücker line coor-

dinates in S_3 [2] [3] as well as the (1:1) correspondence between the lines of S_3 and the points of a non-singular hyperquadric surface in a projective space of five dimensions. Moreover, we assume the reader is familiar with the properties of this correspondence as found in both books [2] [4] given in our references.

In developing the table of classification and in presenting the new line involution we use certain fundamental definitions and the symbols m, i, j, add k to denote the characteristics of line involutions. Some of these definitions and the meaning of the characteristics of a line involution we present as follows:

1. The Plücker coordinates or simply coordinates of a line, l, in S_3 and given by a 6-tuple $(x_1, x_2, x_3, x_4, x_5, x_6)$ if and only if

$$x_1 x_6 - x_2 x_5 + x_3 x_4 = 0;$$

2. a complex of lines in S_3 is a set of lines whose coordinates satisfy one algebraic equation;

3. a congruence of lines is a set of lines, in S_3, whose coordinates satisfy two independent algebraic equations;

4. a regulus of lines is a set of lines, in S_3, whose coordinates satisfy three independent equations;

5. a line involution T is a transformation of lines such that $T^2(\ell) = \ell$ for all lines ℓ.

The characteristics m, i, j, and k of a line involution are defined as follows:

1. The symbol m denotes the order of the involution, that is, the order of the image regulus of a general pen-

cil of lines in S_3;

2. The symbol i denotes the order of the invariant complex of lines, that is, the order of the complex of those lines which are their own images under the involution;

3. The symbol k denotes the order of the singular complex of lines, that is, the order of the complex of those lines which meet their images;

4. The symbol j denotes the order of the surface determined by the lines which connect corresponding points of a general line and its image curve on the hyperquadric of lines in S_5.

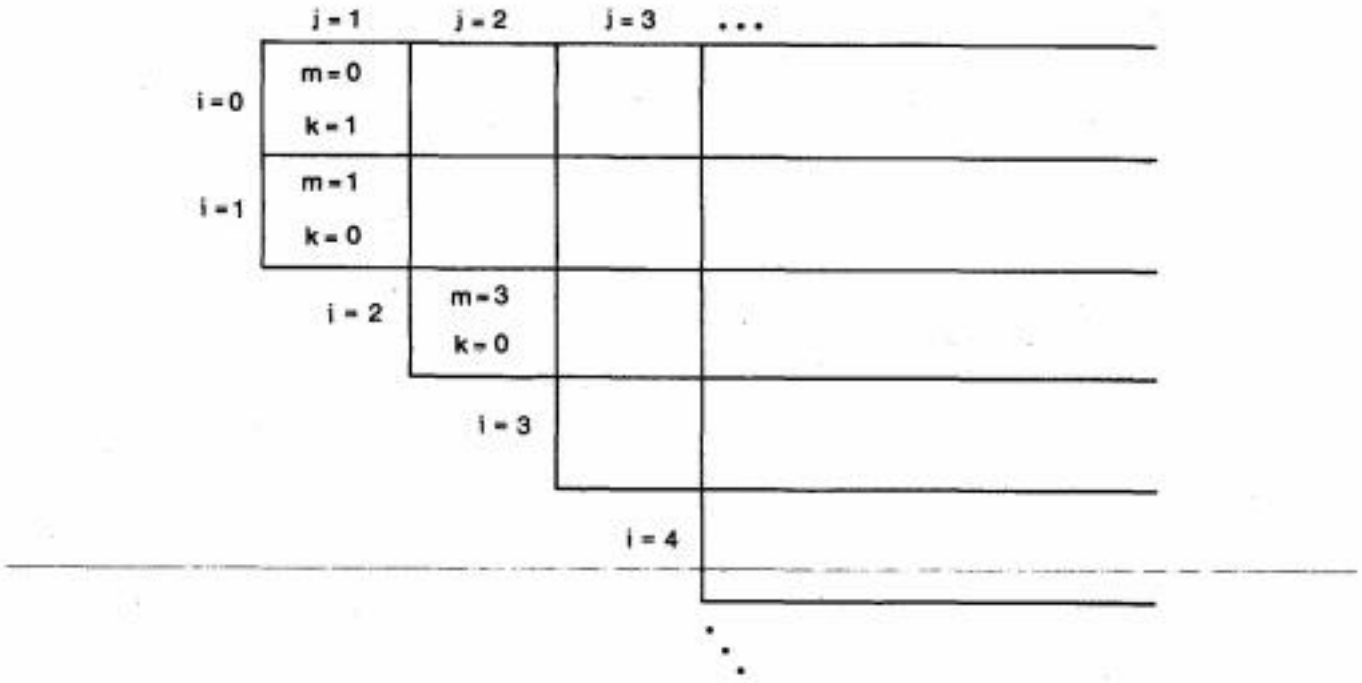
Now, the numbers representing the characteristics of a line involution satisfy the relations [7]

$$\text{I. } m+1 = 2j-k = j+i,$$

and for different values of i=0,1,2,... and j=1,2,3... we find corresponding values for m and k as the following classification table indicates.

The series of involutions discussed by Clarkson [1]and Stratopoulos [6] as well as the twisted-cubic line involution presented in this paper exhibit the existence of involutions of all orders for which i=0. Isolated examples of involutions for which i=1 can be found throughout the literature of the subject [9]. Furthermore, there is no reason to believe that involutions of all orders having i=1 will not be discovered. Finally, a paper published by C.R. Wylie [8] establishes the existence of line involutions of order m=2r+1 and i=r+1 where $r \geq 1$. Thus, the existence of line involutions for each cell of the above table has been established.

CLASSIFICATION TABLE

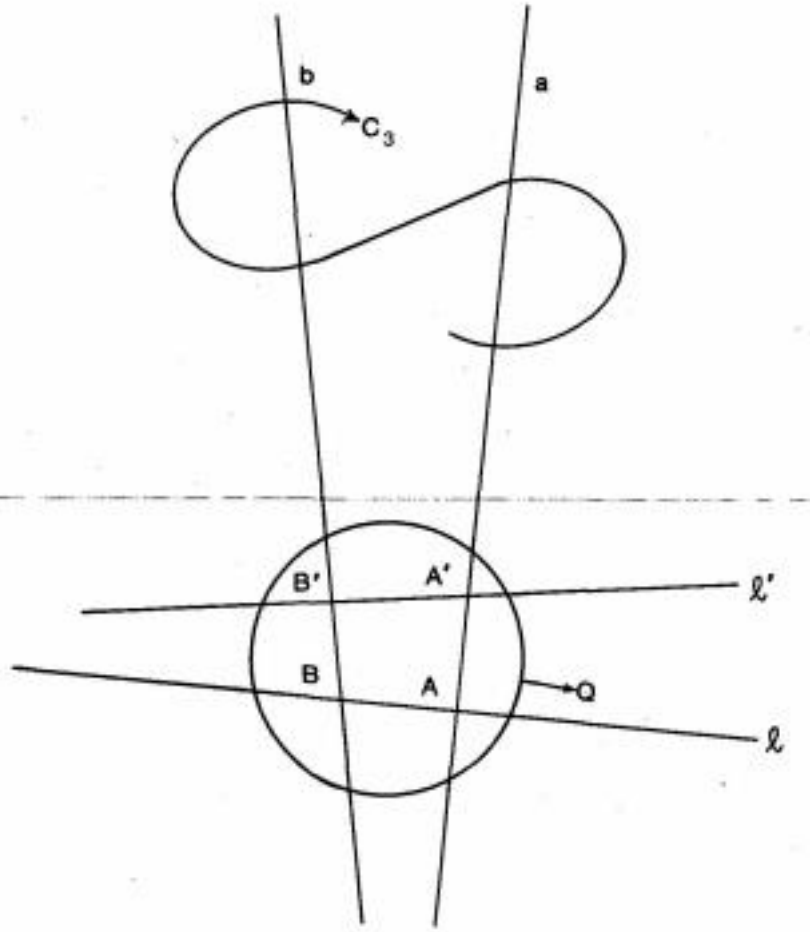


In attempting the total classification of involutions, we are interested to know whether a given set m, i, j, k is adequate to characterize the involution uniquely. Few examples of uniqueness of such involutions are found in papers published by Stratopoulos and Wylie [7] [10], and again there is no reason to expect that other uniqueness problems concerning line involutions cannot be established. It our hope that this paper will stimulate further research into the area of existence, uniqueness and classification of line involutions.

The twisted-cubic line involution:

Let Q be a non-singular quadric surface and C_3 a twisted-cubic curve in S_3, then any line ℓ meets Q in two points A and B (see figure) and from each of these points we can

draw one and only one transversal of C_3. Thus, let a and b be the transversals of C_3 drawn from the points A and B respectively.



Let A' and B' be the second intersections of the transversals a and b and the quadric Q, respectively. The line ℓ', determined by the points A' and B' is the image of the line ℓ under the line involution T ($T^2(\ell)=\ell$) specified by the quadric Q and the twisted-cubic.

In investigating this line involution we shall denote by r a typical generator of the R-regulus of Q and by g a typical generator of the G-regulus of the quadric Q. Now, under the point involution induced on Q every generator of either regulus of Q has an image-curve on Q. For instance,

if C is the image of an r-generator on Q, then the plane determined by r and any g-generator meets the twisted - cubic in three points K, L, and M. The lines KL, KM, and IM interesect g and are transversals of C_3 drawn from points of r. Therefore the generator g meets the image-curve, C, of r at 3 points. Since the congruence of transversals of any two r and $\bar{r}$ of the R-regulus has class one and order one (i.e. it is a (1,1)) and since the transversals of C_3 form a (3,1) congruence (class 3, and order 1), the two congruences have four lines in common. Thus, $\bar{r}$ meets the image-curve C of r at four points and, the image-curve has characteristics 4 and 3 or it is a (4,3) curve on Q. Likewise, we find that the image-curve C' of any g-generator is a (3,4) curve on Q.

Furthermore, any conic $C_2(1,1)$ on Q intersects the curve C (4,3) at seven points whose images must be on the image, C_2', of the conic and r. Hence, r intersects C_2' (the image of the conic) at seven points. Similarly g intersects C_2' at seven points. Therefore, the image-curve of a general conic under the point involution induced on Q is a C_2 (7,7) curve.

The order of the line involution.

Let p be a pencil of lines and Pp the plane of this pencil. Then Pp intersects the quadric Q in a conic C_2. Let I be the image regulus of the pencil p under the line involution; and let C'(7,7) be the image of the conic, C_2, under the point involution on Q. Then every line, ℓ, of p meets the conic, C_2, at two points, A and A_1, and the correspondence A to A_1 is a (1:1) correspondence on the conic. Furthermore, the image ℓ', of the line ℓ meets the curve C' at the points A' and A_1' which are images of A and A_1 under the point involution. Thus, the (1:1) correspondence, A to A_1, on C_2 induces a (1:1) correspondence, A' to A'_1, on the curve C'.

Since the images, ℓ', of the lines of the pencil, p, connect corresponding points of a (1:1) correspondence on the points of C', and since C' is rational, because it is in (1:1) correspondence with a conic, it follows that these lines, ℓ', form a V_2^{13} surface [2] which must be the image of the pencil p. Hence the order, m, of the line involution is 13.

The invariant complex.

For a line ℓ to be invariant, under this involution, the intersections A and B of the line and the quadric Q must coincide with the second intersections A' and B' of the transversals of C_3 drawn from the points A and B, respectively. That is, a line ℓ is invariant if and only if either meets twice the locus of points at which the transversals of C_3 are tangent to Q or if ℓ is a secant of C_3. Since the secants of C_3 form a congruence of lines, the secants of C_3 constitute a (3,1) - congruence of invariant lines. On the other hand, since the locus of tangencies on Q of transversals of C is a curve, C_t, and since the lines meeting C_t twice are also invariant, we obtain a second congruence of invariant lines, namely the congruence of secants of C_t. This investigation leads to the conclusion that there is no complex of invariant lines, and consequently the order, i, of such a complex is zero. By the way, since the secants of C_3 form a (3,1) - congruence and since the lines tangent to Q along a generator of either regulus is a (1,1) - congruence, the two congruences have four lines in common which means a generator of either regulus of Q meets C_t at four points. Thus, the curve C_t has characteristics (4,4) on Q. Consequently, the congruence of invariant lines which are secants, of C_t has class

$$\binom{8}{2} = \frac{8!}{(8-2)!2!} = \frac{7.8}{2} = 28$$

and order $\frac{4^2+4^2-4-4}{2} = 12$ [5]

The singular complex.

Let ℓ be a general line meeting the quadric, Q at the points A and B. Let ℓ' be the image of ℓ under the line involution. Moreover, let A' and B' be the images of the points A and B on Q. Then ℓ' contains the points A' and B', and the lines AA' and BB' are the transversals of the twisted-cubic, C_3, from the points A and B, respectively (see Fig.). Now if the line meets its image ℓ' at a point O, then the plane, P, determined by ℓ and ℓ' contains the point O, as well as the transversals AA' and BB'. Consequently, P contains four points of C_3 namely the points of C_3 on the transversals AA' and BB'. This would provide us with a contradiction unless

a. The transversals AA' and BB' meet in a point of C_3;
or b. the intersection, O, of ℓ and ℓ' is on Q and coincides with one or the other of the intersections, A and B of ℓ and Q.

In investigating these possibilities, let us assume first that the transversals, AA' and BB', meet in a point V of C_3. Then, the line ℓ belongs to the congruence of secants of the curve which is the intersection of Q with the cone whose vertex is the point V and whose directrix is the curve C_3. Since there exist ∞^1 points on C_3, and since each of these points determines a cone with the curve C_3, it follows that we must have ∞^1 of these congruences. Now, since every line in one of these congruences is clearly singular, these ∞^1 congruences form a complex of singular lines.

On the other hand, if the intersection, O, of the line

ℓ and its image ℓ' coincides with the intersection, say A, of ℓ and Q, the transversal AA' of C_3 from A is a tangent line to Q at A. Of course, the same holds true concerning the other transversal BB' whenever the point O coincides with the intersection, B, of ℓ and Q. Thus, the singular lines in this case, meet the locus of points at which the transversals of C_3 are tangent to Q. The locus of these points of tangency is a (4,4) - curve, C_t, on Q. Consequently, since every line meeting C_t at a single point is clearly singular, it follows that C_t is the directrix of a complex of singular lines. Moreover, since 8 lines of a general pencil meet the curve C_t at a single point, the order of this complex must be 8. Thus, the complex of singular lines is composite, the components being the two complexes obtained in the above investigation. Now, since the order of the involution is m=13, since the order of the complex of invariant lines is i=0, and since

$$m+1 = 2j-k = j+i,$$

we obtain k=j=14 or that the order of the composite complex of singular lines is 14. Since the order of one of its components is eight, the order of the other must be six. Clearly, this line involution belongs in the cell of the classification table with i=0 and j=14 or in the first row and 14th column of the table.

The images of a plane-field and a bundle.

Let p be a general plane-field of lines in S_3. Then p meets the quadric Q in a conic C_2, and every line of p intersects C_2 in two points. However, the conic C_2 is transformed into a curve C of characteristics (7,7) on Q. Thus, the lines of p are transformed into the congruence of se-

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