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LUSIN SETS AND GENERALIZED SPECTRA OF TOPOLOGICAL ALGEBRAS

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1. Introduction

In this note we are concerned with generalized spectra of topological algebras, in relation with *K*-analytic sets and in particular with Lusin sets (see below for definitions).

Let E and F be topological algebras and let *M*(E,F) be the generalized spectrum of E (for given F) (i.e. the set of non-zero continous algebra homomorphisms of E into F) topologized as a subset of L_s (E,F) where L_s (E,F) is the space of continous linear maps between the topological vector spaces E and F, equipped with the topology of simple convergence in E. If E and F are unital algebras, then the elements of *M* (E,F) are assumed to be identity preserving ([11]).

K-analytic sets or *K*-Souslin sets (every *K*-analytic set is *K*-Souslin) is a class of sets larger that Borel sets, with important properties in their applications to Integration and Radon measure theory.

We know that every Lusin or Souslin space (they belong to the class of *K*-Souslin spaces) is a Radon measure. Also, we note that the measurability on these spaces is obtained on spaces which are not locally compact. This is very useful because some of the function spaces in probability theory are not locally compact ([9]).

For basic definitions and theorems on *K*-Souslin sets see:

[1, chap. 3], [2, ch. IX], [4, I], [8, II], [10, App.]

We shall show that:

i) If E is a unital locally m-convex algebra, countable strict inductive limit of separable Fréchet algebras, E_n, $n \in \mathbb{N}$ such that $(E_n)_{n \in \mathbb{N}}$ to be closed subspaces of E, the generalized spectrum *M* (E, V), where V is a unital locally m-convex Hausdorff algebra, countable union of continous linear images of separable Fréchet algebras, is a Lusin space.

ii) Let $(E_i)_{i \in \mathbb{N}}$ be unital locally m-convex separable Fréchet algebras. Let V as in (i). Then, the generalized spectrum $\mathcal{M}(\underset{i \in \mathbb{N}}{\otimes} E_i, V)$ where $\underset{i \in \mathbb{N}}{\otimes} E_i$ is the countable projective topological tensorial procuct of $(E_i)_{i \in \mathbb{N}}$ is a Lusin space.

2. Preliminaries

2.1. By a topological (resp. locally convex) algebra we mean an algebra E equipped with a Hausdorff topology such that the (underlying) vector space is a topological (resp. locally convex) vector space and the multiplication in E is separately continous (considered as a bilinear map of ExE into E).

A particular case of locally convex algebras, used in the following, are the locally m-convex ones. Such an algebra has a (jointly) continuous multiplication ([7]).

2.2. A topological space is said to be polish if it is separable, if there is a distance on the space compatible with the topology and the space is complete. Let $X[\tau]$ be a Hausdorff topological space. $X[\tau]$ is said to be Souslin (resp. Lusin) if it is the continous (resp. and injective) image of a polish space.

3. On the Inductive Limit

3.1. Let $(E_\alpha, f_{\alpha\beta})$ be an inductive limit of l.c. algebras and $E = \varinjlim E_\alpha$ the corresponding locally convex inductive limit topological algebra.

For a given locally convex topological algebra V we have the relation ([11, p. 179]).

$$L_s(E, V) = L_s(\varinjlim E_\alpha, V) = \varprojlim L_s(E_\alpha, V)$$

(within an homeomorphism)

where $\varprojlim L_s(E_\alpha, V)$ is the projective limit of $L_s(E_\alpha, V)$ ([6, p. 95, 96]). If in particular, we have the following conditions:

(i′) $\forall (\alpha, \beta) \in I \times I$ and $g \in \mathcal{M}(E_\alpha, V)$ we have $\mathrm{Im}(f_{\beta\alpha}) \cap \complement \ker(g) \neq \emptyset$

(ii′) $\forall a \in I$ and $h \in \mathcal{M}(E_\alpha, V)$ we have $\mathrm{Im}(f_\alpha) \cap \complement \ker h \neq \emptyset$

then, we shall have: ([11])

$$M(E, V) = M(\varinjlim E_\alpha, V) = \varprojlim M(E_\alpha, V)$$

(within an homeomorphism)

3.2. Let E be the inductive limit of a sequence of l.c. separable Fréshet spaces $(E_n)_{n \in \mathbb{N}}$, $E = \varinjlim E_n$ such that

(i'') $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence

(ii'') Every compact subset of E is compact subset of E_n, for some $n \in \mathbb{N}$. An example of a inductive limit of a sequence of l.c. spaces $(E_n)_{n \in \mathbb{N}}$ $E = \varinjlim E_n$. which satisfies the conditions (i''), (ii'') is every strict inductive limit E of a sequence $(E_n)_{n \in \mathbb{N}}$ of Hausdorff and closed subspaces of E ([3, p. 144, Prop. 6, p. 152)].

Proposition 3.3. Let E be a unital locally m-convex algebra, countable inductive limit of separable Fréchet algebras which satisfies the conditions (i''), (ii''), and V as in (i). Then, the generalized spectrum $M(E, V)$ is a Lusin space.

Proof: It is known that the space $L_c(E_n, V)$ (i.e. the space $L(E_n, C)$ endowed with the compact-open topology) is a Lusin space. Also $L_s(E_n, V)$ is a Lusin space ([8,p. 111]). Since,

$$L(E, V) = L_s(\varinjlim E_n, V) = \varprojlim L_s(E_n, V),\ n \in \mathbb{N}$$

the space $L_s(E, V)$ is a Lusin space (ibid, ch. II). By assumptions of E and V, $M(E, V)$ is a Lusin set.

In the following we demostrate the existence of Souslin subset in the space $C_s(X, E)$:

Proposition 3.4. Let X be a locally compact Souslin space and E be a topological vector Souslin space. Then, there is a Souslin subspace of $C_s(X, E)$.

Proof: Let $K(X, \mathbb{R})$ be the space of continous (real) functions with compact support. Since $K(X, \mathbb{R})$, is the inductive limit of $K_k(X, \mathbb{R})$, $\varinjlim K_k(X, \mathbb{R}) = K(X, \mathbb{R})$ where $K_k(X, R)$ is the space of continous linear functions with support a subset of K, for every K compact subset of E and endowed with the topology of compact convergence ([4, p. 65, p. 301]). For this topology the spaces K_k (X, $\mathbb{R}$) are polish spaces and theirs inductive limit is Souslin ([4, p. 301], [8, p. 111]). We consider the map

$$h: K_{\mathbb{R}}(X) \times E \longrightarrow C_s(X,E) \; h(g,f) = \ell_{g,f}, \; \ell_{g,f}(x) = g(x)f, \; x \in X, \; f \in E$$

The map h is continous and the space h ($K_{\mathbb{R}}$ (X) xE) is a Souslin subspace of C_s (X, E).

4. On the Tensor Product Algebra

Theorem 4.1. Let E and F be unital l.m-convex separable Fréchet algebras and V as in proposition 3.3. Then, $M(E \underset{\pi}{\otimes} F, V)$ where π is projective tensorial topology (compatible topology) on $E \otimes F$ ([11, p. 178], ([10, p. 434]), is a Lusin space.

Proof: We define the map ψ: $M(E \underset{\pi}{\otimes} F, V) \longrightarrow M(E, V) \times M(F, V)$ by the relation:

$$\psi(h) = (f,g)_h \text{ with } \quad \begin{aligned} f(x) &= h(x \otimes 1), \; x \in E \\ g(y) &= h(1 \otimes y), \; y \in F \end{aligned} \quad h \in M(E \otimes F, V)$$

Let $Q = \{ (f, g) : f(x)\, g(y) = g(y)\, f(x), \; (x, y) \in E \times F \}$
Q is a closed subset of M (E, V) x M (F, V). The map ψ has as image the set Q and it is an homeomorphism ([11, p. 179]). By well known properties of Lusin sets, $M(E \underset{\pi}{\otimes} F, V)$ is a Lusin space.

Now, if we define the concept of the *l*. convex, not finite topological tensorial procuct we can state a result which constitutes an extended form of Theorem 4.1. For the definition see [11, Th. 5.2].

Proposition 4.2. Let V and $(E_i)_{i \in \mathbb{N}}$ be topological algebras with the corresponding assumptions of 4.2. Theorem. Then, the spectrum $M(\underset{i \in \mathbb{N}}{\otimes} E_i, V)$ is a Lusin space.

Proof: We define the map ψ: $M(\underset{i \in \mathbb{N}}{\otimes}, V) \longrightarrow \varprojlim Q_\alpha \subseteq \prod_{i \in \mathbb{N}} M(E_i, V)$ which is bicontinous and onto ([11]). Since the spaces $Q_\alpha = M_s(E_\alpha, V)_{,\alpha \in \mathbb{N}}$ finite, are Lusin spaces, and the family $\{Q_\alpha\}_{\alpha \in \mathbb{N}}$ is countable ([5, Vol. I, p. 16]) the result is obvious.

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