Authors: P. N. Agrawal, Vijay Gupta

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SIMULTANEOUS APPROXIMATION BY LINEAR COMBINATION OF MODIFIED BERNSTEIN POLYNOMIALS

BY

P.N. AGRAWAL AND VIJAY GUPTA

1. Introduction

Durrmeyer [2] introduced the following integral modification of Bernstein polynomials as

$$M_n(f;x)=(n+1)\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_0^1 p_{n,\nu}(t)f(t)\,dt,\quad f\in L_1(0,1)$$

where

$$p_{n,\nu}(x)=\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu},\qquad \nu=0,1,2,\ldots,n.$$

May [4] and Rathore [5] have described a method for forming linear combinations of positive linear operators to improve the order of approximation. The approximation process is a follows:

$$M_n(f,k,x)=\sum_{j=0}^{k} C(j,k)M_{d_j n}(f;x),$$

where $d_o,d_1,d_2,\ldots,d_k$ are $k+1$ arbitrary but fixed distinct positive integers and

$$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i}\ ,\quad k\neq 0\quad\text{and}\quad C(0,0)=1.$$

In the sequel, $L_B(0,1)$ denotes the class of bounded and Lebesgue integrable functions on $[0,1]$ and $\|\cdot\|$ the sup norm on $[a,b]\subset(0,1)$.

The object of this paper is to prove the following results:

Theorem 1.1. If $f\ \ L_B(0,1)$ admits a derivative of order $2k+r+2$ at a point $x\ \ [0,1]$, then

$$\lim_{n\to\infty} n^{k+1}[M^{(r)}(f,k,x)-f^{(r)}(x)]=\sum_{i=r}^{2k+r+2} f^{(i)}(x)Q(i,k,r,x) \tag{1.1}$$

and

$$\lim_{n\to\infty} n^{k+1}[M^{(r)}(f,k+1,x)-f^{(r)}(x)]=0, \tag{1.2}$$

where $Q(i,k,r,x)$ are certain polynomials in x.
Further, (1.1) and (1.2) hold uniformly on $[0,1]$ if $f^{(2k+r+2)}\in C[0,1]$.

Theorem 1.2. Let $1\leqslant p\leqslant 2k+2$, $f\in L_B(0,1)$ and $n>0$ be arbitrary. If $f^{(p+r)}$ exists and is continuous on $(a-\eta,\ b+\eta)\subset(0,1)$ having the modulus of continuity $\omega_{f^{(p+r)}}(\delta)$ on $(a-\eta,\ b+\eta)$ then, for n sufficiently large

$$\|M^{(r)}(f,k,x)-f^{(r)}(x)\|\leqslant \mathrm{Max}\{C_1 n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}),\ C_2 n^{-(k+1)}\}, \tag{1.3}$$

where $C_1=C_1(k,p,r)$, $C_2=C_2(k,p,r,f)$.

2. Auxiliary results

We require the following results:

Lemma 2.1[1]. For the function $T_{n,m}$ (m being nonnegative integer) defined on $[0,1]$ as

$$T_{n,m}(x)=\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)(x-t)^m dt,$$

we have

(a) $T_{n,m}(x)$ is a polynomial in x of degree m;

(b) $T_{n,m}(x)=O(n^{-([(m+1)/2]+1)})$, where $[\beta]$ denotes the integral part of β;

(c) $T_{n,2m}(x)$ is a polynomial in $x(1-x)$ and is uniformly equivalent to

$$\frac{1}{n^{m+1}}\frac{(2m)!}{(m)!}\{x(1-x)\}^m,\quad n\to\infty;$$

(d) $T_{n,2m-1}(x)$ is a polynomial in $x(1-x)$ multiplied by $(1-2x)$ and is uniformly equivalent to

$$-\frac{1}{n^{m+1}}\frac{(2m)!}{2(m-1)!}(1-2x)\{x(1-x)\}^{m-1},\quad n\to\infty.$$

Lemma 2.2[6]. There exist polynomials $q_{i,j,r}(x)$ independent of n and k such that

$$\frac{d^r}{dx^r}x^k(1-x)^{n-k}=\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^j(-nx)^j q_{i,j,r}(x)\,x^{k-r}(1-x)^{n-k-r}.$$

Proposition 2.3[3]. For $p \in N$, $M_n((\cdot - x)^p, k, x) = n^{-(k+1)} \cdot \{Q(p,k,x) + o(1)\}$ where $Q(p,k,x)$ are certain polynomials in x of degree at most p.

Theorem 2.4[3]. Let $f \in L_B(0,1)$. If $f^{(2k+2)}$ exists at a point $x \in [0,1]$, then

$$\lim_{n\to\infty} n^{k+1}[M_n(f,k,x) - f(x)] = \sum_{j=1}^{2k+2} \frac{f^{(j)}(x)}{(j)!}\, Q(j,k,x) \tag{2.1}$$

and

$$\lim_{n\to\infty} n^{k+1}[M_n(f,k+1,x) - f(x)] = 0 \tag{2.2}$$

where $Q(j,k,x)$ are certain polynomials in x of degree at most j. Moreover, if $f^{(2k+2)} \in C[0,1]$, then (2.1) and (2.2) hold uniformly on $[0,1]$.

3. Proofs

Proof of theorem 1.1. By the Taylor's expansion of $f(t)$, we have

$$f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!}\,(t-x)^i + g(t-x)(t-x)^{2k+r+2}, \tag{3.1}$$

where $g(t-x) \to 0$ as $t \to x$ and is bounded and integrable function on $[-x, 1-x]$. Using (3.1) we have

$$\begin{aligned}
M^{(r)}(f,k,x) &= \sum_{j=0}^{k} C(j,k) M_{d_jn}^{(r)}(f;x) \\
&= \sum_{j=0}^{k} C(j,k)(d_jn+1) \sum_{v=0}^{d_jn} p_{d_jn,v}^{(r)}(x) \int_0^1 p_{d_jn,v}(t) f(t)\,dt \\
&= \sum_{j=0}^{k} C(j,k)(d_jn+1) \sum_{v=0}^{d_jn} p_{d_jn,v}^{(r)}(x) \int_0^1 p_{d_jn,v}(t). \\
&\quad \Big\{ \sum_{j=0}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!}\,(t-x)^i + g(t-x)(t-x)^{2k+r+2} \Big\}\, dt \\
&= \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!} \sum_{j=0}^{k} C(j,k) M_{d_jn}^{(r)}((\cdot - x)^i; x) \\
&\quad + \sum_{j=0}^{k} C(j,k) M_{d_jn}^{(r)}(g(t-x)(t-x)^{2k+r+2}; x).
\end{aligned} \tag{3.2}$$

Now, with $D \equiv \frac{d}{dx}$ by Lemma 2.1(a) and (2.1) we obtain

$$I_1 = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!} \sum_{j=0}^{k} C(j,k) M_{djn}^{(r)}((\cdot - x)^i; x) =$$

$$= \sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!} \sum_{j=0}^{k} C(j,k) M_{djn}^{(r)}((\cdot - x)^i; x) =$$

$$= \sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!} \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} x^{i-l} M_n^{(r)}(t^l; k, x) =$$

$$= \sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!} \cdot \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} x^{i-l} \cdot$$

$$\cdot [D^r x^l + n^{-(k+1)} \{ \sum_{j=1}^{2k+2} D^r \left[\frac{Q(j,k,x)}{(j)!} D^j x^l \right] + o(1) \}]$$

$$= \sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{(i)!} (r)! \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} \binom{l}{r} x^{l-r}$$

$$+ n^{-(k+1)} \sum_{i=r}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x) + o(n^{-(k+1)})$$

$$= f^{(r)}(x) + n^{-(k+1)} \sum_{i=r}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x) + o(n^{-(k+1)}),$$

using the identities

$$\sum_{l=0}^{i} (-1)^l \binom{i}{l} \binom{l}{r} = \begin{cases} 0, & i > r \\ (-1)^r, & i = r \end{cases}.$$

To prove the first assertion, from (3.2) and the estimate of I_1 it is sufficient to show that

$$n^{k+1} M_n^{(r)}(g(t-x)(t-x)^{2k+r+2}; x) \to 0 \text{ as } n \to \infty$$

Now

$$M_n^{(r)}(g(t-x)(t-x)^{2k+r+2}; x)$$

$$= (n+1) \sum_{v=0}^{n} p_{n,v}^{(r)}(x) \int_0^1 p_{n,v}(t) g(t-x)(t-x)^{2k+r+2} dt.$$

Suppose

$$I_2 = (n+1) \sum_{v=0}^{n} x^r (1-x)^r p_{n,v}^{(r)}(x) \int_0^1 p_{n,v}(t) g(t-x)(t-x)^{2k+r+2} dt.$$

Using Lemma 2.2 we get

$$I_2=(n+1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i q_{i,j,r}(x)\sum_{\nu=0}^{n} p_{n,\nu}(x)(\nu-nx)^j\cdot$$
$$\cdot\int_0^1 p_{n,\nu}(t)g(t-x)(t-x)^{2k+r+2}dt.$$

Put $M=\sup_{x\in[0,1]}\sup_{2i+j\le r}|q_{i,j,r}(x)|$ and then apply Schwarz inequality for summation to have

$$|I_2|\le M(n+1)\sum_{2i+j\le r} n^i\Big\{\sum_{\nu=0}^{n}(\nu-nx)^{2j}p_{n,\nu}(x)\Big\}^{1/2}\cdot$$
$$\cdot\Big\{\sum_{\nu=0}^{n}p_{n,\nu}(x)\Big(\int_0^1 p_{n,\nu}(t)g(t-x)(t-x)^{2k+r+2}dt\Big)^2\Big\}^{1/2}. \qquad (3.3)$$

By the known property of Lorentz [6]

$$\frac{1}{n^j}\sup_{x\in[0,1]}\Big\{\sum_{\nu=0}^{n}(\nu-nx)^{2j}p_{n,\nu}(x)\Big\}$$

is bounded for all values of j.

For a given $\varepsilon>0$, there exists a $\delta>0$ such that $|g(t-x)|<\varepsilon$ whenever $|t-x|<\delta$, and for $|t-x|\ge\delta$ the boundedness of $g(t-x)$ implies that

$$|g(t-x)|\le K(|t-x|/\delta),\ \text{where } K=\sup_{u\in[-x,1-x]}|g(u)| \qquad (3.4)$$

Hence, $(g(t-x))^2\le\varepsilon^2+(t-x)^2\dfrac{K^2}{\delta^2}$.

Now, using Schwarz inequality for integral and (3.4) we have

$$\Big(\int_0^1 p_{n,\nu}(t)g(t-x)(t-x)^{2k+r+2}dt\Big)^2\le\frac{1}{(n+1)}\cdot$$
$$\cdot\int_0^1 p_{n,\nu}(t)(g(t-x))^2(t-x)^{2(2k+r+2)}dt<\frac{1}{(n+1)}\cdot$$
$$\cdot\int_0^1 p_{n,\nu}(t)\Big\{\varepsilon^2+\frac{K^2}{\delta^2}(t-x)^2\Big\}(t-x)^{2(2k+r+2)}dt.$$

Applying Lemma 2.1(b) we obtain

$$\sum_{\nu=0}^{n}p_{n,\nu}(x)\Big(\int_0^1 p_{n,\nu}(t)g(t-x)(t-x)^{2k+r+2}dt\Big)^2<\sum_{\nu=0}^{n}p_{n,\nu}(x)\frac{1}{(n+1)}\cdot$$

$$\cdot \int_0^1 p_{n,\nu}(t)\left\{\varepsilon^2+\frac{K^2}{\delta^2}(t-x)^2\right\}(t-x)^{2(2k+r+2)}dt$$

$$=\frac{1}{(n+1)}\left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_0^1 p_{n,\nu}(t)\varepsilon^2(t-x)^{2(2k+r+2)}dt\right.$$

$$\left.+\frac{K^2}{\delta^2}\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_0^1 p_{n,\nu}(t)(t-x)^{2(2k+r+3)}dt\right\}$$

$$=\frac{1}{(n+1)}\left\{\varepsilon^2\cdot O(n^{-(2k+r+3)})+\frac{K^2}{\delta^2}\,O(n^{-(2k+r+4)}\right\}$$

$$=\frac{1}{(n+1)}\,\varepsilon^2\cdot O(n^{-(2k+r+3)})=\varepsilon^2 O\left(\frac{1}{n^{2k+r+4}}\right).$$

Thus, finally in (3.3) we have

$$|I_2|\leqslant M(n+1)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} C_3 n^{i+(j/2)}\cdot\varepsilon\cdot C_4 n^{-1/2(2k+r+4)}=\varepsilon O(n^{-(k+1)}.$$

Due to arbitrariness of $\varepsilon>0$, $n^{k+1}I_2\to 0$ as $n\to\infty$.

The assertion (1.2) can be proved along similar lines by noting the fact that $M_n((\cdot-x)^i, k+1, x))=O(n^{-(k+2)})$, $i=1,2,3\ldots$ which follows from proposition 2.3.

The last assertion is due to the uniform continuity of $f^{(2k+r+2)}$ on $[0,1]$ (enabling δ to become independent of $x\in[0,1]$) and the uniformity of the $o(n^{-(k+1)})$ term in the estimate of I_1 (because, in fact, it is a polynomial in x).

This completes the proof.

PROOF OF THEOREM 1.2. For $x\in[a,b]$, by the hypothesis, we have

$$f(t)=\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{(i)!}(t-x)^i+\frac{\{f^{(p+r)}(\xi)-f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r}$$

$$+h(t,x)\chi(t),\quad t\in[0,1],\tag{3.5}$$

where ξ lies between t and x, and $\chi(t)$ is the characteristic function of the set $[0,1]\setminus(a-\eta,b+\eta)$.

The function $h(t,x)$ for $x\in[a,b]$ is bounded by $M|t-x|^{p+r}$ for some constant M. Operating on this equality by $M^{(r)}(\cdot,k,x)$ and breaking the right hand side into three parts I_1, I_2 and I_3 say, corresponding to the three terms on the right hand side of (3.5), as in the proof of Theorem 1.1, we have

$$I_1 = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{(i)!} M_n^{(r)}((\cdot - x)^i, k, x) = f^{(r)}(x) + O(n^{-(k+1)}),$$

uniformly in $x \in [a,b]$.

Now to estimate I_2, we have for every $\delta > 0$

$$|f^{(p+r)}(\xi) - f^{(p+r)}(x)| \leq \omega_{f^{(p+r)}}(|\xi - x|) \leq \omega_{f^{(p+r)}}(|t-x|)$$

$$\leq 1 + \frac{|t-x|}{\delta} \omega_{f^{(p+r)}}(\delta). \tag{3.6}$$

Since,

$$I_2 = \sum_{i=0}^{k} C(j,k)(d_j n + 1) \sum_{v=0}^{d_j n} p_{d_j n,v}^{(r)}(x) \cdot$$

$$\cdot \int_0^1 p_{d_j n,v}(t) \frac{\{f^{(p+r)}(\xi) - f^{(p+r)}(x)\}}{(p+r)!} (t-x)^{p+r} dt,$$

using (3.6) and Lemma 2.2 we get

$$|I_2| \leq \frac{1}{(p+r)!} \sum_{i=0}^{k} |C(j,k)|(d_j n + 1) \sum_{v=0}^{d_j n} |p_{d_j n,v}^{(r)}(x)| \cdot$$

$$\cdot \int_0^1 p_{d_j n,v}(t) \left(1 + \frac{|t-x|}{\delta}\right) |t-x)^{p+r} \omega_{f^{(p+r)}}(\delta) dt$$

$$\leq \frac{\omega_{f^{(p+r)}}(\delta)}{(p+r)!} \sum_{j=0}^{k} |C(j,k)|(d_j n + 1) \sum_{\substack{2i+s \leq r \\ i,s \geq 0}} (d_j n)^i \frac{|q_{i,s,r}(x)|}{x^r (1-x)^r} \cdot$$

$$\cdot \sum_{v=0}^{d_j n} p_{d_j n,v}(x) |(v - d_j n x)|^s \cdot$$

$$\cdot \int_0^1 p_{d_j n,v}(t) [|t-x|^{p+r} + \frac{1}{\delta} |t-x|^{p+r+1}] dt.$$

Putting $K = \sup_{x \in [a,b]} \sup_{\substack{2i+s \leq r \\ i,s \geq 0}} \frac{|q_{i,s,r}(x)|}{x^r (1-x)^r}$, then applying Schwarz inequalities for summation and for integral and Lemma 2.1(b), as in the proof of Theorem 1.1, we have

$$|I_2| \leq K[C_5 n^{-p/2} + \frac{1}{\delta} C_6 n^{-(p+1)/2}] \omega_{f^{(p+r)}}(\delta).$$

Now by choosing $\delta = n^{-1/2}$, it follows that

$$I_2 = \omega_{f^{(p+r)}}(n^{-1/2})O(n^{-p/2}),$$

where the O-term holds uniformly in $x \in [a,b]$.
Finally, choosing a positive number δ such that $|t-x| \geqslant \delta$ we get

$$|I_3| \leqslant \sum_{j=0}^{k} |C(j,k)|(d_j n+1) \sum_{\substack{2i+s\leqslant r \\ i,s\geqslant 0}} (d_j n)^i \frac{|q_{i,s,r}(x)|}{x^r(1-x)^r} \cdot$$
$$\cdot \sum_{\nu=0}^{d_j n} p_{d_j n,\nu}(x)|(\nu - d_j nx)|^s \int_{[0,1]\setminus(a-\eta,b+\eta)} p_{d_j n,\nu}(t) M|t-x|^{p+r}dt$$
$$\leqslant M \sum_{j=0}^{k} |C(j,k)|(d_j n+1) \sum_{\substack{2i+s\leqslant r \\ i,s\geqslant 0}} (d_j n)^i \frac{|q_{i,s,r}(x)|}{x^r(1-x)^r} \cdot$$
$$\cdot \sum_{\nu=0}^{d_j n} p_{d_j n,\nu}(x)|(\nu - d_j nx)|^s \int_0^1 p_{d_j n,\nu}(t) \frac{(t-x)^{2m}}{\delta^{2m-p-r}} dt.$$

Thus,

$$I_3 = O(n^{(r-2m)/2}), \quad m > \frac{2k+r+2}{2}.$$

So, $I_3 = o(n^{-(k+1)})$ uniformly in x $[a,b]$.
The theorem follows from the estimates of I_1, I_2 and I_3.

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Department of Mathematics
University of Roorkee
Roorkee-247667 (U.P.) INDIA