Authors: G. S. Srivastava, O. P. Juneja

Title: On the maximum term and central index of dirichlet series representing analytic function in a half plane

Creator: HDML

BULLETIN THE
GREEK MATHEMATICAL SOCIETY
VOLUME 30, 1989

ON THE MAXIMUM TERM AND CENTRAL INDEX OF DIRICHLET SERIES REPRESENTING ANALYTIC FUNCTION IN A HALF PLANE

BY

G.S. SRIVASTAVA AND O.P. JUNEJA

1. Consider the Dirichlet series

$$\sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \tag{1.1}$$

where $s=\sigma+it$; σ and t real variables, $0\leqslant\lambda_1<\lambda_2<\ldots$, $\lim_{n\to\infty}\lambda_n=\infty$ and

$$\limsup_{n\to\infty}\frac{n}{\lambda_n}=D<\infty \tag{1.2}$$

If C and A denote the abscissa of convergence and abscissa of absolute convergence respectively, of the series in (1.1) then since (1.2) is satisfied, we have from [4, p. 166]

$$C=A=\liminf_{n\to\infty}\frac{\log|a_n|^{-1}}{\lambda_n} \tag{1.3}$$

It is well known that the series (1.1) represents an analytic function f(s) in the half plane $\sigma<A$. For this function f(s), let us put

$$M(\sigma)=\operatorname*{l\cdot u\cdot b}_{-\infty<t<\infty}|f(\sigma+it)|,\quad m(\sigma)=\max_{n\geqslant 1}\{|a_n|e^{\sigma\lambda_n}\}$$

and $$\lambda_{N(\sigma)}=\max_{n\geqslant 1}\{\lambda_n: m(\sigma)=|a_n|e^{\sigma\lambda_n}\}.$$

$M(\sigma)$ is called the maximum modulus, $m(\sigma)$ the maximum term of f(s) and $\lambda_{N(\sigma)}$ is called the central index of the maximum term for Real $s=\sigma$. It is known [6] that $\log M(\sigma)$ is an increasing convex function of σ for $\sigma<A$, $m(\sigma)\leqslant M(\sigma)$. From the definitions, it also follows that $m(\sigma)$ and $\lambda_{N(\sigma)}$ are nondecreasing functions of σ. Further the maximum term and central index are connected by the relation ([5, Lemma], [7]):

$$\log m(\sigma)=\log m(\sigma_1)+\int_{\sigma_1}^{\sigma}\lambda_{N(u)}du,\quad -\infty<\sigma_1<\sigma<A. \tag{1.4}$$

If $C=A=\infty$, then f(s), the sum function of (1.1) represents an entire function and in this case, much work has been done to study relative growth of $M(\sigma)$, $m(\sigma)$, $\lambda_{N(\sigma)}$ etc. by introducing the concept of order, type, growth numbers etc. However, if f is analytic in the half plane $\sigma<A<\infty$, then the above concepts have been introduced comparatively recently; [see e.g. 1,2,3, 5 etc.].

For a real number P, we define $\log^+P=\log P$ if $P\geqslant 1$ and $\log^+P=0$ if $0<P\leqslant 1$. Then the order ρ of f(s) is defined as [2]

$$\limsup_{\sigma\to A}\frac{\log^+\log^+M(\sigma)}{-\log(1-e^{\sigma-A})}=\rho,\quad 0\leqslant\rho\leqslant\infty \tag{1.5}$$

If the order ρ of f(s) satisfies $0<\rho<\infty$, then we define type T and lower type τ of f(s) as [2]

$$\limsup_{\sigma\to A}\inf\frac{\log M(\sigma)}{(1-e^{\sigma-A})^{-\rho}}=\frac{T}{\tau},\quad 0\leqslant\tau\leqslant T\leqslant\infty \tag{1.6}$$

The function f(s) is said to be of **regular growth** if limit exists on left hand side of (1.5). Further, f(s) is said to be of **perfectly regular growth** if $0<\rho<\infty$ and $0<\tau=T<\infty$.

Suppose that the exponents $\{\lambda_n\}$ in (1.1) satisfy the condition

$$\liminf_{n\to\infty}(\lambda_n-\lambda_{n-1})=\beta>0 \tag{1.7}$$

Then for $0<\rho<\infty$, it was proved [5, Theorem 2] that

$$\limsup_{\sigma\to A}\frac{\log^+\log^+m(\sigma)}{-\log(1-e^{\sigma-A})}=\rho. \tag{1.8}$$

By using [5, Theorem 1], we can easily prove that for $0<\rho<\infty$,

$$\limsup_{\sigma\to A}\inf\frac{\log^+m(\sigma)}{(1-e^{\sigma-A})^{-\rho}}=\frac{T}{\tau}. \tag{1.9}$$

It is easily seen that for $0<\rho<\infty$, from (1.8) we have $\log m(\sigma)\to\infty$ as $\sigma\to A$. Hence we omit the use of $\log^+m(\sigma)$ and write simply $\log m(\sigma)$. Nandan [5, Theorem 3] proved that for $\rho>0$ and under the condition (1.7), we have

$$\limsup_{\sigma\to A}\frac{\log\lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}=1+\rho \tag{1.10}$$

Hence we find that $\lambda_{N(\sigma)}\to\infty$ as $\sigma\to A$. Juneja and Handan obtained the

following coefficient characterization for the order ρ [2, Theorem 1], viz.,

$$\frac{\rho}{\rho+1} = \limsup_{n\to\infty} \frac{\log^+\{\log|a_n| + A\lambda_n\}}{\log\lambda_n} \tag{1.11}$$

where the left hand side is to be interpreted as 1 when $\rho=\infty$.

The aim of this paper is to obtain for the analytic function f(s), a characterization for the type T and lower type τ in terms of the maximum term and central index. In the latter part, we obtain some relations which throw light on the relative asymptotic growths of $m(\sigma)$ and $\lambda_{N(\sigma)}$.

2. In the subsequent analysis, we shall assume throughout that f(s) as given by (1.1) is analytic in the half plane $\sigma<A$ and the exponents $\{\lambda_n\}$ satisfy the condition (1.7). The parameters ρ, T and τ have the same meaning as in (1.5) and (1.6). We shall also assume that $0<\rho<\infty$ so that $\log m(\sigma)\to\infty$ and $\lambda_{N(\sigma)}\to\infty$ as $\sigma\to A$ and (1.9) holds. Further we shall assume that the type T satisfies $0\leqslant T<\infty$.

We now prove

Theorem 1. We have

$$\rho T = \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \limsup_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1 + \frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}, \tag{2.1}$$

where $x = 1/(1-e^{\sigma-A})$.

Proof. For any $u\geqslant 0$ and $\rho>0$, we have

$$\rho u < \left(\frac{\rho}{\rho+1}\right)^{\rho+1} (1+u)^{\rho+1}.$$

Hence

$$\rho\,\frac{\log m(\sigma)}{x^\rho} < \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1 + \frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}. \tag{2.2}$$

On proceeding to limits as $\sigma\to A$, we obtain

$$\rho T \leqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \limsup_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \; 1 + \left[\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}.$$

To obtain the reverse inequality, we consider the relation (1.4). Then for $-\infty<y$, $\sigma<A$, we have

$$\log m(\sigma) = \log m(y) + \int_y^{\sigma} \lambda_{N(u)}\, du.$$

Since $\lambda_{N(\sigma)}$ is non-decreasing, we have

$$\frac{\log m(\sigma)}{x^{\rho}} \geqslant \frac{\log m(y) + (\sigma - y)\lambda_{N(y)}}{x^{\rho}}. \tag{2.4}$$

We consider the values of $\sigma = \sigma_y$ given by the roots of the equation

$$\rho\sigma - e^{A-\sigma} = \frac{\rho y \lambda_{N(y)} - \rho \log m(y) - \lambda_{N(y)}}{\lambda_{N(y)}}, \tag{2.5}$$

for $y < A$. It is easy to see that as $\sigma \to A$, $e^{A-\sigma} \to 1 + A - \sigma$. Hence for σ near to A, we can write $e^{A-\sigma} = 1 + (A - \sigma)\ \varepsilon(\sigma)$, where $\varepsilon(\sigma) \to 1$ as $\sigma \to A$. Consequently from (2.5) we have

$$\sigma_y = \frac{[A\,\varepsilon(y) + \rho y]\lambda_{N(y)} - \rho \log m(y)}{[\rho + \varepsilon(y)]\lambda_{N(y)}}$$

where $\varepsilon(y) \to 1$ as $y \to A$. From the above expression it can be easily seen that $\sigma_y < A$. Also from (1.11), we have for any arbitrary $\delta > 0$ and all $n > n_0(\delta)$,

$$\log |a_n| + A\lambda_n < (\lambda_n)^{\rho/(1+\rho)-\delta},$$

or

$$\log m(\sigma) < (\lambda_n)^{\rho/(1+\rho)-\delta}.$$

Since $0 < \rho/(1+\rho) < 1$, we get

$$\lim_{\sigma \to A} \frac{\log m(\sigma)}{\lambda_{N(\sigma)}} = 0.$$

Hence as $y \to A$, $\sigma_y \to A$. Now for $\sigma = \sigma_y$ we have

$$\log m(y) + (\sigma - y)\lambda_{N(y)} = \frac{\varepsilon(y)}{\rho + \varepsilon(y)}\ [\log m(y) + (A - y)\lambda_{N(y)}].$$

Consequently from (2.4) we have for $\sigma = \sigma_y$,

$$\frac{\log m(\sigma)}{x^{\rho}} \geqslant \frac{\varepsilon(y)}{\rho + \varepsilon(y)} \left[\frac{\log m(y) + (A - y)\lambda_{N(y)}}{x^{\rho}} \right].$$

Now $x^{-1} = 1 - e^{\sigma_y - A} = (A - \sigma_y)\delta(y)$, $\delta(y) \to 1$ as $y \to A$

$$= \frac{\rho\delta(y)}{\rho + \varepsilon(y)} \left[A - y + \frac{\log m(y)}{\lambda_{N(y)}} \right].$$

Hence we have

$$\frac{\log m(\sigma)}{x^{\rho}} \geqslant \frac{\varepsilon(y)(\delta(y))^{\rho}\rho^{\rho}}{[\rho+\varepsilon(y)]^{\rho+1}} \cdot (A-y)^{\rho+1}\lambda_{N(y)}\left[1+\frac{\log m(y)}{(A-y)\lambda_{N(y)}}\right]^{\rho+1}.$$

Lastly, as $y \to A$, $\sigma_y \to A$, $A-y \to 1-e^{y-A}$, $\varepsilon(y) \to 1$ and $\delta(y) \to 1$. Hence proceeding to limits as $y \to A$, we get

$$\rho T \geqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \limsup_{\sigma \to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}}\left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}, \quad (2.6)$$

where now $x = 1/(1-e^{\sigma-A})$. Combining (2.3) and (2.6), we obtain (2.1). This proves Theorem 1.

The result of Theorem 1 does not hold if lim sup is replaced by lim inf on both sides of (2.1). In fact we have analytic functions for which $T \neq \tau$ although limit on the right hand side of (2.1) exists.

Consider the function $f(s) = \sum_{n=1}^{\infty} \exp(2^n)\exp(4^n s)$. Then it can be easily seen that (1.7) and hence (1.2) is satisfied with $\lambda_n = 4^n$. Consequently, $f(s)$ is analytic for $\sigma < 0$, as verified from (1.3). Using (1.11), we find that the order of $f(s)$ is 1.

Also, from (1.9), we find that $T = 1/4$ and $\tau = 2/9$. However, for $\rho = 1$ and $A = 0$, we have for this function

$$\left(\frac{\rho}{\rho+1}\right)^{\rho+1} \lim_{\sigma \to 0} \frac{\lambda_{N(\sigma)}}{x^2}\left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1} = \frac{1}{4},$$

where $x = 1/(1-e^{\sigma})$. Thus Theorem 1 does not hold for lower type τ.

The upper bound for the lower type τ can be immediately obtained from (2.2). Thus we have

$$\rho\tau \leqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \cdot \liminf_{\sigma \to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}}\left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}.$$

In the next few theorems, we obtain some lower bounds for the value of τ as well as sufficient conditions under which (2.1) holds for lower type τ. We prove

Theorem 2. Let $f(s) = \sum_{k=1}^{\infty} \exp(s\lambda_n)$ be analytic for $\sigma < A$, having order $\rho (0 < \rho < \infty)$ and lower type τ. Then for any increasing sequence $\{\sigma_n\}$ of real numbers such that $\sigma_n \to A$, as $n \to \infty$, we have

$$\rho\tau \geqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \cdot \liminf_{n\to\infty} \frac{\lambda_{N(\sigma_{n-1})}}{(x_n)^{\rho+1}} \left[1 + \frac{x_n \log m(\sigma_n)}{\lambda_{N(\sigma_n)}}\right]^{\rho+1}$$

where $x_n = 1/(1-e^{\sigma_n - A})$.

Proof. Let us put

$$\liminf_{n\to\infty} \frac{\lambda_{N(\sigma_{n-1})}}{(x_n)^{\rho+1}} \left[1 + \frac{x_n \log m(\sigma_n)}{\lambda_{N(s_n)}}\right]^{r+1} = q.$$

Then (2.8) obviously holds if $q = 0$. Hence let $q > 0$ and choose , $0 < \varepsilon < q$. Then for $n > n_0(\varepsilon)$, we have

$$\frac{\lambda_{N(\sigma_{n-1})}}{(x_n)^{\rho+1}} \left[1 + \frac{x_n \log m(\sigma_n)}{\lambda_{N(\sigma_n)}}\right]^{\rho+1} > q - \varepsilon,$$

or, since $1 - e^{\sigma - A} \sim A - \sigma$ as $\sigma \to A$, we get

$$\log m(\sigma_n) > \lambda_{N(\sigma_n)} \left[\left\{\frac{q-\varepsilon}{\lambda_{N(\sigma_{n-1})}}\right\}^{1/(\rho+1)} - (A - \sigma_n)\right].$$

Now from (1.4) we have

$$\begin{aligned} \log m(\sigma) &= \log m(\sigma_n) + \int_{\sigma_n}^{\sigma} \lambda_{N(u)} du \\ &\geqslant \log m(\sigma_n) + \lambda_{N(\sigma_n)}(\sigma - \sigma_n) \\ &> \lambda_{N(\sigma_n)} \left[\left\{\frac{q-\varepsilon}{\lambda_{N(\sigma_{n-1})}}\right\}^{1/(\rho+1)} - (A - \sigma)\right]. \end{aligned}$$

Let $\{P_n\}$ be a sequence of real numbers defined by

$$P_n = A - \frac{\rho}{\rho+1} \left\{\frac{q-\varepsilon}{\lambda_{N(\sigma_{n-1})}}\right\}^{1/(\rho+1)}, \quad n > n_0.$$

Then for $n \to \infty$, $\sigma_n \to A$ and consequently $P_n \to A$, since we have $\lambda_{N(\sigma)} \to \infty$ as $\sigma \to A$. Now if σ satisfies $P_n \leqslant \sigma < P_{n+1}$, then for $x = 1/(1-e^{\sigma - A})$,

$$\frac{\log m(\sigma)}{x^\rho} > \frac{\lambda_{N(\sigma_n)}}{x^\rho} \left[\left\{\frac{q-\varepsilon}{\lambda_{N(\sigma_{n-1})}}\right\}^{1/\rho+1} - (A - \sigma)\right]$$

$$> \frac{\lambda_{N(\sigma_n)}}{(1-e^{P_{n+1}-A})^{\cdot\rho}} \qquad \left[(A - P_n)\left(\frac{\rho+1}{\rho}\right) - (A - P_n)\right]$$

$$= \frac{(\varepsilon(n))^{\rho}}{\rho}(A-P_n)(A-P_{n+1})^{\rho}\lambda_{N(\sigma_n)}$$

where $\varepsilon(n) \to 1$ as $n \to \infty$, $P_n \to A$. Now substituting for P_{n+1}, we have

$$\frac{\log m(\sigma)}{x^{\rho}} > \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \frac{(q-\varepsilon)(\varepsilon(n))^{\rho}(A-P_n)}{\rho(A-P_{n+1})}, \quad n > n_0.$$

Evidently $(A-P_n)/(A-P_{n+1}) \geqslant 1$. Hence on proceeding to limits as $n \to \infty$, $\sigma \to A$, we obtain

$$\rho\tau \geqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} q,$$

since $\varepsilon > 0$ was arbitrary. This proves (2.8).

Using elementary properties of lim inf etc., and using the assumption $T < \infty$ together with (2.1), we can prove the following corollaries to Theorem 2.

Corollary 1. We have

$$\rho\tau \geqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \cdot \max_{\{\sigma_n\}} \left[\liminf_{n\to\infty} \frac{\lambda_{N(\sigma_{n-1})}}{(x_n)^{\rho+1}} \left\{1 + \frac{x_n \log m(\sigma_n)}{\lambda_{N(\sigma_n)}}\right\}^{\rho+1}\right]$$

where $x_n = 1/1-e^{\sigma_n - A})$ and maximum on the right hand side is taken over all increasing sequences $\{\sigma_n\}$ of real numbers such that $\lim_{n\to\infty} \sigma_n = A$.

Corollary 2. We have

$$\rho\tau \geqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \cdot \max_{\{\sigma_n\}} \left[\liminf_{n\to\infty} \frac{\lambda_{N(\sigma_{n-1})}}{\lambda_{N(\sigma_n)}}\right] \times$$

$$\liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left\{1 + \frac{x \log m(\sigma)}{\lambda_{N(\sigma)}}\right\}^{\rho+1},$$

where $x = 1/(1-e^{\sigma - A})$.

Now we prove

Theorem 3. If f(s) satisfies conditions of Theorem 2 and there exists an increasing sequence of real numbers $\{\sigma_n\}$, $\lim \sigma_n = A$, such that

$$\lambda_{N(\sigma_{n-1})} \simeq \lambda_{N(\sigma_n)} \text{ as } n \to \infty, \tag{2.9}$$

then

$$\rho\tau = \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left\{1 + \frac{x \log m(\sigma)}{\lambda_{N(\sigma)}}\right\}^{\rho+1}. \quad (2.10)$$

Proof. Using the given condition (2.9) in Corollary 2, we obtain

$$\rho\tau \geqslant \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1 + \frac{x \log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}.$$

The result now follows on combining the above inequality with (2.7).

We now consider two examples, one for which (2.9) is satisfied and then other for which (2.9) is not satisfied.

EXAMPLE 1. Let $f(s) = \sum \exp(n)\exp(n^2 s)$, $s = \sigma + it$. Then (1.7) is satisfied and on using (1.3) it follows that f is analytic for $\sigma < 0$. Also, from the coefficient formula (1.11), we get

$$\frac{\rho}{1+\rho} = \lim_{n\to\infty} \frac{\log^+[\log^+|a_n| + A\lambda_n]}{\log\lambda_n} = \frac{1}{2}$$

i.e. $\rho = 1$. It can be easily seen that for $-(2n-1)^{-1} \leqslant \sigma < -(2n+1)^{-1}$, $m(\sigma) = e^{(n+n^2\sigma)}$ and $\lambda_{N(\sigma)} = n^2$. Hence for the sequence $\sigma_n = -(2n-1)^{-1}$, the condition (2.9) is satisfied. Evidently for this function f, $T = \tau = \frac{1}{4}$ and (2.10) is also satisfied.

EXAMPLE 2. Let $f(s) = \sum \exp(2^n)\exp(4^n s)$, $s = \sigma + it$. As shown earlier, f(s) is analytic for $\sigma < 0$, is of order $\rho = 1$ and type $T = 1/4$, lower type $\tau = 2/9$. For $\frac{1}{3\times 2^{n-1}} \leqslant \sigma < -\frac{1}{3\times 2^n}$, $m(\sigma) = \exp(2^n + 4^n\sigma)$ and $\lambda_{N(\sigma)} = 4^n$. As shown earlier,

$$\left(\frac{\rho}{\rho+1}\right)^{\rho+1} \lim_{\sigma\to 0} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1 + \frac{x \log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1} = \frac{1}{4},$$

and (2.10) does not hold. It is also easy to see that (2.9) does not hold for any sequence $\{\sigma_n\}$, $\sigma_n \to 0$.

Whether the holding of condition (2.9) is necessary for validity of (2.10) is still an open question. Before we obtain necessary and sufficient condition for f to be of perfectly regular growth, we prove

Theorem 4. Let f(s) be analytic for $\sigma<A$, of order ρ, $0<\rho<\infty$ and of perfectly regular growth. Then

$$\lim_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \qquad \text{exists.}$$

Proof. Since f(s) is of perfectly regular growth, we have from (1.9), for any arbitrary $\varepsilon>0$ and all σ, $\sigma_0<\sigma<A$,

$$T-\varepsilon<\frac{\log m(\sigma)}{x^{\rho}}<T+\varepsilon,$$

where $x=1/(1-e^{\sigma-A})$. From (1.4), for $\sigma_0<\sigma<u<A$, since $\lambda_{N(\sigma)}$ is non-decreasing, we have

$$(u-\sigma)\lambda_{N(\sigma)}=\int_{\sigma}^{u}\lambda_{N(\sigma)}dy\leqslant\int_{\sigma}^{u}\lambda_{N(y)}dy=\log m(u)-\log m(\sigma)$$
$$<\frac{(T+\varepsilon)}{(1-e^{u-A})^{\rho}}-\frac{(T-\varepsilon)}{(1-e^{\sigma-A})^{\rho}}.$$

Let us choose $u=A+\log\left[1-\frac{1-e^{\sigma-A}}{\eta}\right]$ where $\eta>1$. Then $\sigma<u<A$ and $u\to A$ as $\sigma\to A$. Hence

$$\frac{\lambda_{N(\sigma)}}{(1-e^{\sigma-A})^{\rho+1}}<\frac{(1-e^{\sigma-A})}{(u-\sigma)}\left[(T+\varepsilon)\left\{\frac{1-e^{\sigma-A}}{1-e^{u-A}}\right\}^{\rho}-(T-\varepsilon)\right].$$

It can be easily seen that $\frac{1-e^{\sigma-A}}{1-e^{u-A}}=\eta$ and $\lim_{\sigma\to A}\frac{1-e^{\sigma-A}}{u-\sigma}=\frac{\eta}{\eta-1}$.

Hence on proceeding to limits as $\sigma\to A$, we obtain, since $\varepsilon>0$ was arbitrary,

$$\limsup_{\sigma\to A}\frac{\lambda_{N(\sigma)}}{(1-e^{\sigma-A})^{\rho+1}}\leqslant T\eta\,\frac{(\eta^{\rho}-1)}{\eta-1}.$$

Again we have

$$(u-\sigma)\lambda_{N(u)}\geqslant\int_{\sigma}^{u}\lambda_{N(y)}dy=\log m(u)-\log m(\sigma)>$$

$$\frac{(T-\varepsilon)}{(1-e^{u-A})^{\rho}}-\frac{(T+\varepsilon)}{(1-e^{\sigma-A})^{\rho}},\quad u>\sigma>\sigma_0.$$

Hence

$$\frac{\lambda_{N(u)}}{(1-e^{u-A})^{\rho+1}}>\frac{(1-e^{u-A})}{u-\sigma}\left[(T-\varepsilon)-(T+\varepsilon)\left\{\frac{1-e^{u-A}}{1-e^{\sigma-A}}\right\}^{\rho}\right]$$

or, $\liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{(1-e^{\sigma\cdot A})^{-\rho-1}} \geqslant \frac{T(\eta^\rho-1)}{\eta^\rho(\eta-1)}$.

Since the above inequalities hold for arbitrary $\eta>1$, we get on proceeding to limits as $\eta\to 1^+$,

$$\rho T \leqslant \liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{(1-e^{\sigma\cdot A})^{-\rho-1}} \leqslant \limsup_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{(1-e^{\sigma\cdot A})^{-\rho-1}} \leqslant \rho T$$

in view of the relation $\lim_{\eta\to 1^+} \frac{\eta^\rho-1}{\eta-1}=\rho$. Thus we get

$$\liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{(1-e^{\sigma\cdot A})^{-\rho-1}} = \limsup_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{(1-e^{\sigma\cdot A})^{-\rho-1}} = \rho T$$

and Theorem 4 follows.

Lastly we prove

Theorem 5. A necessary and sufficiently condition for $f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ to be of perfectly regular growth, of order ρ and type T, $0<T<\infty$, is that the condition (2.9) should be satisfied and

$$\lim_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1} = \left(\frac{\rho+1}{\rho}\right)^{\rho+1}(\rho T). \qquad (2.11)$$

Proof. Let $f(s)$ be of perfectly regular growth. Then from (2.1) and (2.7), we have

$$\limsup_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1} = \left(\frac{\rho+1}{\rho}\right)^{\rho+1}(\rho T) = \left(\frac{\rho+1}{\rho}\right)^{\rho+1}(\rho\tau)$$

$$\leqslant \liminf_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1},$$

$$\text{i.e. } \lim_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1+\frac{x\log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1} = \left(\frac{\rho+1}{\rho}\right)^{\rho+1}(\rho T).$$

Further, from Theorem 4, we also have

$$\lim_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{(1-e^{\sigma\cdot A})^{-\rho-1}} = d \quad \text{(say)} \qquad (2.12)$$

exists. Now from Theorem 5 [3], applied for the case $p=2$, we have, since limit in (2.12) exists;

$$1 \leqslant \liminf_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} \leqslant \limsup_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} \leqslant 1.$$

Hence we have $\lambda_n \simeq \lambda_{n+1}$ as $n \to \infty$. Thus for any increasing sequence $\{\sigma_n\}$ of real numbers with $\sigma_n \to A$ as $n \to \infty$, we obviously have $\lambda_{N(\sigma_{n-1})} \simeq \lambda_{N(\sigma_n)}$. Thus condition (2.9) holds.

Conversly suppose that (2.9) holds for some increasing sequence $\{\sigma_n\}$ and limit in (2.11) exists. Then from (2.1) and (2.10) it follows that

$$\rho\tau = \rho T = \left(\frac{\rho}{\rho+1}\right)^{\rho+1} \lim_{\sigma\to A} \frac{\lambda_{N(\sigma)}}{x^{\rho+1}} \left[1 + \frac{x \log m(\sigma)}{\lambda_{N(\sigma)}}\right]^{\rho+1}.$$

Hence $\tau = T$ and $f(s)$ is of perfectly regular growth. This proves Theorem 5.

The authors are extremely thankful to the referee for his critical comments and valuable suggestions which helped in improving this paper.

REFERENCES

1. E. Dagene, The central index of a Dirichlet series, Litovsk Mat. Sb. **8** (1968), 503-522.
2. O.P. Juneja and K. Nandan, On the growth of analytic functions represented by Dirichlet series, Portugal. Math. **37** (1978), 125-133.
3. Naresh Kumar and G.S. Srivastava, On the coefficients of analytic functions of fast growth, Bull. Math. Soc. Sci. Math. R.S. Roumanie, **32** (80 no.3 (1988), 243-250.
4. S. Mandelbrojt, "Dirichlet Series", Rice Institute Pamphlet, **31** (1944), 157-272.
5. K. Nandan, On the maximum term and maximum modulus of analytic functions represented by Dirichlet series, Annal. Polon. Math., **28** (1973), 213-222.
6. J.F. Ritt, On certain points in the theory of Dirichlet series, Amer. J. Math. **50** (1928), 73-86.
7. C.Y. Yu, Sur les droites de Borel de certaines fonctiones entieres, Ann. Sci. Ecole Norm. Sup. (3) **68**, (1951), 65-104.

Department of mathematics
University of Roorkee
Roorkee, India

Department of Mathematics
Indian Institute of Technology
Kanpur, India