Author: Aggeliki Voudouri

Title: On the distribution of the present value of a continuous uniform cash flow

Creator: HDML

BULLETIN THE
GREEK MATHEMATICAL SOCIETY
VOLUME 30, 1989

ON THE DISTRIBUTION OF THE PRESENT VALUE OF A CONTINUOUS UNIFORM CASH FLOW

BY

AGGELIKI VOUDOURI

1. Introduction

Let X represent a continuous uniform cash flow that starts at time zero and ends at the future time T. Its present value Y is given by

$$Y = \frac{X}{r}\left[1 - e^{-rT}\right] \tag{1.1}$$

where r is the nominal continuous-compound interest rate. The work of Young and Contreras [9] and Rosenthal [7] extends the applicability of present value models to decision problems under uncertainty. Their work provides formulas for the expected value and variance of the present value Y when X and T are independent random variables.

The present paper is devoted to the study of the distribution of Y when X and T are independent random variables. Conditions are given for the distribution of the present value Y to be imbedded in certain classes of distributions.

In recent related contributions to the field of stochastic extensions of present value models, Artikis and Voudouri [1] have considered the case where X is a single random cash flow to be paid at some future random time T. Artikis and Voudouri have shown that the distribution of the present value Y of X belongs to the class of α-unimodal distributions if T is exponentially distributed. The class of α-unimodal distributions contains the class of ordinary unimodal distributions, Olsen and Savage [6]. The concept of α-unimodality deals with the distribution function of the product of two independent random variables, one of which has a beta (α, 1) distribution.

Artikis and Jerwood [3] have extended Artikis and Voudouri's present value model to consider cases where the random variable T is a geometric random sum. Artikis and Jerwood's present value model has interesting applications in risk management, replacement policies and capital budgeting.

In this paper, it is shown that the distribution of the present value Y of a continuous uniform cash flow X with duration T is a member of the class of ν-unimodal distributions if the duration T is exponentially-distributed. The class of ν-unimodal distributions is also an extension of the class of ordinary unimodal distributions, Gomes and Pestana [4].

The results given in the present paper show promise of simplifying certain calculations and modeling difficulties in financial decision making.

2. Arbitrarily - Distributed Timing

Let T be a continuous nonnegative random variable with probability density function $f_T(t)$. Consider the random variable

$$W = 1 - e^{-rT}, \qquad r > 0.$$

The probability density function $f_W(w)$ of W is easily demonstrated to be

$$f_W(w) = \frac{1}{r(1-w)} f_T\left(-\frac{1}{r}\log(1-w)\right), \qquad 0 < w < 1 \qquad (2.1)$$

Let X be a continuous nonnegative random variable with characteristic function $\phi_X(u)$. If X and T are independent random variables then

$$\phi_Y(u) = \int_0^1 \phi_X(uw/r) \cdot \frac{1}{r(1-w)} f_T\left(-\frac{1}{r}\log(1-w)dw\right) \qquad (2.2)$$

is the characteristic function of the random variable

$$Y = \frac{X}{r}[1 - e^{-rT}], \qquad (2.3)$$

which denotes the present value of a continuous uniform cash flow. From (2.2) it follows that the distribution of Y is very complicated when the duration T of a continuous uniform cash flow is arbitrarily-distributed.

3. Exponentially - Distributed Timing

We suppose that T is exponentially-distributed with probability density function

$$f_T(t) = \lambda e^{-\lambda t}, \qquad t > 0, \qquad \lambda > 0$$

then from (2.1) and (2.2) if follows that the characteristic function of Y is given by

$$\phi_Y(u) = \frac{\lambda}{r}\int_0^1 \phi_X(uw/r)(1-w)^{\frac{\lambda}{r}-1}dw \tag{3.1}$$

characteristic functions of the form (3.1) belong to the class of ν-unimodal characteristic functions with $\nu=\lambda/r$, Gomes and Pestana [4]. The concept of ν-unimodality deals with the distribution function of two independent random variables, one of which has a beta (1, ν) distribution. Gomes and Pestana's ν-unimodality is an extension of ordinary unimodality which is a concept with theoretical and practical interest, Khinchine [5]. Properties of ν-unimodal distributions have been established by Artikis and Voudouri [2]. From the fact that the exponential distribution is the most commonly encountered timing distribution it follows that the class of ν-unimodal distributions in present value problems of continuous uniform cash flows is the rule than the exception. We may also infer that it is the factor

$$W = 1 - e^{-rT},$$

with T exponentially-distributed, which is primarily responsible for the introdustion of the ν-unimodality into the distribution of the present value Y in (2.3).

We consider two particular cases of (3.1). For $\lambda=r$ in (3.1) we get

$$\phi_Y(u) = \int_0^1 \phi_X(uw/r)\,dw. \tag{3.2}$$

Characteristic functions of the form (3.2) belong to distribution functions having unique mode at zero, Khinchine [5]. For $\lambda=2r$ in (3.1) we get

$$\phi_Y(u) = 2\int_0^1 \phi_X(uw/r)(1-w)\,dw. \tag{3.3}$$

Characteristic functions of the form (3.3) belong to distribution functions having convex probability density functions in the half lines $x>0$ and $x<0$, Sakovic [8]. The class of irdinary unimodal distributions and the class of distributions with convex densities contain many distributions which are very important in probability theory and in mathematical statistics.

From a practical point of view the results of this section suggest possible applications of a wide class of distributions in present value models of a continuous uniform cash flow.

4. Conclusion

The use of expected value and variance for analyzing present values of random cash flow profiles under random timing does not always provide sufficient information for financial decision making. Complete characterizations of the distributions of such present values provide management with sufficient information upon which to base a financial decision. In this paper we investigate the distribution of the present value of continuous uniform cash flows X with random duration T, such as profits that start upon completion of construction of a plant or profits that cease upon failure of a piece of equipment. In cases where the duration T is exponentially distributed and independent of the continuous uniform cash flow X, we evaluate the distribution of the present value of X.

The results of the present paper enables management to fully differentiate between competing continuous uniform cash flows with exponentially - distributed durations.

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