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SURFACES OF GENERAL TYPE IN P^4

BY

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Abstract. In this paper we show that a non-singular surface of general type with canonical divisor very ample and $p_g = 5$ under the canonical embedding it is a complete intersection in P^4.

1. Introduction

Let S be a non-singular irreducible surface of general type with canonical divisor K very ample, i.e. K induces an embedding $S \subset P^{p_g - 1}$ where p_g is the geometric genus of S. In our paper [6] we studied such surfaces with $p_g \leqslant 7$. In the case $p_g = 5$, in particular, we showed that S is a complete intersection in P^4 with one exception where $\deg S = 12$, $\chi(S) = 5$ and $q = 1$.
In this note we show that this last case does not occur by showing that the existence of such a surface would imply the existence of an indecomposable rank 2 bundle on P^4 with Chern classes $c_1 = 0$ and $c_2 = 3$.

I would like to thank Lawrence Ein for pointing out to me W. Barth's and G. Elencwajg's paper [1] where they show that such bundles do not exist on P^4.

2. The result

Theorem. Let S be a non-singular irreducible surface in P^4 with canonical divisor K very ample and $S \subset P^4$ the canonical embedding. Then S is a complete intersection in P^4.

Proof. Let $d = \deg(S) = K^2$. By Hartshorne ([4] p. 434) we have

$$d^2 - 10d - 5HK - 2K^2 + 12\chi(S) = 0$$

and in our case $H = K$. Therefore we get

$$d(17 - d) = 12\chi(S).$$

The only positive integral solutions are

$$\chi(S)=6 \text{ and } d=8 \text{ or } 9, \text{ or}$$
$$\chi(S)=5 \text{ and } d=12.$$

In [6] we showed that if $\chi(S)=6$ then S is a complete intersection. Hence we only need to eliminate the case $\chi(S)=5$ and $d=K^2=12$.
Let S be such a surface. By Hartshorne ([5], Proposition 6.1), since $\omega_S \cong O_S(1)$, S is the zero set of a section of a bundle E of rank 2 on P^4. Furthermore $\chi(S)=5$ and $p_g=5$ and $q=1$, therefore S is not a complete intersection which implies that E is indecomposable. Let N_S be the normal bundle on S in P^4. We then have the following:

(1) $c_2(E\otimes O_S)=(S.S)_{P^4}=c_2(N_S)$ and the free resolution

$0\to L\to \overset{\vee}{E}\to I_S\to 0$ where

(2) $L=O_{P^4}(-6)$ by ([5], proof of Prop. 6.1).

Claim. (a) $c_1(E\otimes O_{P^4}(-3))=0$ and

(b) $c_2(E\otimes O_{P^4}(-3))=3$.

Proof of claim. (a) From (2), since E is of rank 2, we have

$$c_1(E\otimes O_{P^4}(-3))=c_1(\check{L}\otimes O_{P^4}(-6))=0.$$

(b) From (1) we only need to show that $c_2(N_S\otimes O_S(-3))=3K^2$.
By ([4] p. 434) $c_1(N_S)=6K$ and $c_2(N_S)=16K^2-c_2(T_S)$. Since $\chi(S)=5$ and $K^2=12$ we have $c_2(T_S)=4K^2$. Therefore $c_2(N_S)=12K^2$ and

$$c_2(N_S\otimes O_S(-3))=c_2(N_S)+c_1(N_S)\cdot(-3K)+9K^2=3K^2.$$

Therefore $E\otimes O_{P^4}(-3)$ is an indecomposable, hence stable [3], algebraic rank 2 bundle on P^4 with $c_1=0$ and $c_2=3$ and by W. Barth and G. Elencwajg ([1], theorem 4.2) such bundles do not exist. q.e.d.

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