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A NOTE ON AUTOMORPHISM GROUP OF AN n-GENERATOR NILPOTENT LIE ALGEBRA

BY

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1. Introduction

In the paper [BG] of R.M. BRYANT and J.R.J. GROVES a construction of an n-generator nilpotent Lie algebra L over an infinite field K (see, also [B]) is given such that the group of linear transformations on L/L' induced by the automorphism group of L is equal to a given Zariski-closed subgroup H of G (i.e., the group of all $n \times n$, $n \geq 2$, invertible matrices with coefficients from K).

To make clear what we are doing here, we shall give a sketch of their construction. First, we need some notation:

F: free Lie algebra over K on n-free generators.
U: polynomial KG-module
$\gamma_i(F)$: thw ith term of the lower central series of F.
$F' = \gamma_2(F)$
$\delta_j(F)$ = the jth term of the derived series of F.
$D^t = D \otimes D \otimes \ldots \otimes D$ t-copies of the determinant polynomial KG-module D.

Let H be a Zariski-closed subgroup of G. Then, there exists a polynomial KG-module U and a subspace V of U such that H is the stabiliser of V in G. Furthermore, there are fully invariant ideals I and J of F, depending upon U with $I \subseteq J \subseteq F'$, F/I is nilpotent such that J/I is central in F/I. Every automorphism of F/I which induces the identity on $(F/I)/(F/I)'$ induces the identity on J/I. Finally, there exists $t \in N$ such that $D^t \otimes U$ is isomorphic as KG-module to a submodule of J/I, say U_*, and V_* is isomorphic as a vector space to $D^t \otimes V$ under the above isomorphism. Denote $E = F/I$. The subspace V_* is contained in $Z(E) \cap E'$, where Z(E) is the centre of E. The desired Lie algebra L has the form E/V_*.

As we have seen, in the construction, two fully invariant ideals I and J of F

play a significant role. The choice of I and J is complicated an depends upon the module U. Dr R.M. Bryant has conjectured if one can choose I and J as successive terms of a sequence of fully invariant ideals of F which is independent upon U. In this note, we give a positive answer to this conjecture. Notation and terminology follow [BG] and [P] when not exlained here. At this point, I would like to thank my supervisor Dr R.M. Bryant for his valuable advice and suggestions. Much of this work have been drawn from the author's Ph. D Thesis [P].

We define a sequence of fully invariant ideals $I_{(i)}$, $i \in \mathbf{N}$, of F, recursively by

$$I_{(1)} = F$$

$$I_{(i+1)} = \left(\sum_{1 \leq j \leq i} (\gamma_j(F') \cap \gamma_{i+2-j}(F))\right) + \delta_3(F).$$

It is easy to verity that $I_{(1)} \supseteq I_{(2)} \supseteq \cdots \supseteq I_{(i)} \cdots$, $i \in \mathbf{N}$, and for each $i \geqslant 1$, $I_{(i)}$ is a fully invariant ideal of F. Furthermore, it is easy to see that $I_{(i)}/I_{(i+1)}$ is central in $F/I_{(i+1)}$ and every automorphism of $F/I_{(i+1)}$ which induces the identity on $(F/I_{(i+1)})/(F/I_{(i+1)})'$ induces the identity on $I_{(m)}/I_{(m+1)}$, all i, $i \in \mathbf{N}$. For each $i \geqslant 1$, F_i denotes the subspace of F spanned by commutators (i.e., Lie products) of length i. let I be a fully invariant ideal of F. Then, $I = \bigoplus_{i \geqslant 1} (I \cap F_i)$

Furthermore, for any fully invariant ideals I and J of F $(I + J) \cap F_r = (I \cap F_r)$, all r N.

If $I \subseteq J$, $J/I = \bigoplus_{i \geqslant 1} ((J \cap F_i) + I)/I$. Let W be a polynomial KG-module. Then, $W = \bigoplus_{i \geqslant 1} W_{s_i}$ where W_{s_i} is a homogeneous polynomial KG-module.

The following Theorem gives answer to R.M. Bryant's conjecture.

Theorem: For any U polynomial KG-module, there exists $m \in \mathbf{N}$ and $t \in \mathbf{N}_o$ such that $D^t \otimes U$ is isomorphic as KG-module to a submodule of $I_{(m)}/I_{(m+1)}$.

It may be proved that the natural number m is a bound of the nilpotency class of the constructed Lie algebra L. This number os enormous. In the author's Ph.D Thesis [P] using a similar techniques as in [BG], we can construct a Lie algebra L which its nilpotency class is bounded by a much smaller function depending upon n, i.e., the number of generators of L/L', and a positive integer r depending upon H. Let U and W be polynomial KG-modules. We write $U \leqslant W$ if there is a submodule of W which is isomorphic to U.

A Proposition has been proved in the author's Ph.D Thesis [P] playing an important role for the proof of the Theorem. For its proof, we use the Littlewood-Richardson rule. Since the application of the rule is so complicated and combersome I prefer not to write the proof down, but to state only the selult. For the details of its proof, we refer to ([P], p.72).

Proposition: For any homogeneous polynomial KG-module W and for all $h \geqslant 2n-1$, $n \geqslant 2$, there exists $t_o \in N_o$ such that for all $t \geqslant t_o$

$$D^t \otimes W \leqslant (\gamma_h(F') + \delta_3(F))/(\gamma_{h+1}(F') + \delta_3(F)).$$

In the Proposition, the natural number h may be proved to be a bound of the nilpotency class of L'. In [P], it has been proved that a nilpotent Lie algebra L may be chosen such that the nilpotency class of L' is bounded by $\min\{2r, 2n-1\}$ for $n \geqslant 3$, ςηερε ρ δεπενδσ θπον H. Φινσλλυ, ιν [P], a 2-generator nilpotent Lie algebra L has been constructed with the desired properties such that the nilpotency class of L' is at most 2.

2. Proof of Theorem

Let U be any polynomial KG-module. Then, $U = W_{t_1} \oplus \cdots \oplus W_{t_i}$ is a homogeneous KG-module of U of degree t_i and $t_1 > t_2 > \cdots > t_r \geqslant 0$. We choose distinct positive integers $h_1, h_2, \ldots, h_r$ such that

$$h_r > h_{r-1} > \cdots > h_1 \geqslant 2n-1 \qquad \text{and}$$

$$h_1 + t_1 = \cdots = h_r + t_r$$

By Proposition, there exist $t_o(h_1), \ldots, t_o(h_r)$ N_o such that for all $t \geqslant \max\{t_o(h_i) : i = 1, \ldots, r\}$

$$D^t \otimes W_{t_i} \leqslant (\gamma_{h_i}(F') + \delta_3(F))/(\gamma_{h_i+1}(F') + \delta_3(F))$$

$i = 1, \ldots, r$. Therefore, we can choose a such t N_o, where $t > 0$ if $t_r = 0$. Let $s_i = nt + t_i$, $i = 1, \ldots, r$. Then, $s_1 + h_1 = \cdots = s_r + h_r$. Write $m = s_i + h_i - 1$, $i = 1, \ldots, r$. Since $D^t \otimes W_{t_i}$ is a homogeneous polynomial KG-module of degree s_i, we obtain

$$D^t \quad W_{t_i} \leqslant ((\gamma_{h_i}(F') + \delta_3(F)) \cap F_{s_i})/((\gamma_{h_i+1}(F') + \delta_3(F)) \cap F_{s_i}) \quad (1)$$

But,

$$I_{(m)} \cap F_{s_i} = \sum_{1 \leq k < m-1} (\gamma_k(F') \cap \gamma_{m+1-k}(F)) \cap F_{s_i} + (\delta_3(F) \cap F_{s_i})$$

Since,

$$\gamma_k(F') \cap \gamma_{m+1-k}(F) \cap F_{s_i} = 0 \text{ if } m+1-k > s_i,$$

$$\gamma_k(F') \cap \gamma_{m+1-k}(F) \cap F_{s_i} = \gamma_k(F') \cap F_{s_i} \text{ if } m+1-k \leqslant s_i$$

or equivalently,

$$\gamma_k(F') \cap \gamma_{m+1-k}(F) \cap F_{s_i} = 0 \text{ if } h_i > k,$$

$$\gamma_k(F') \cap \gamma_{m+1-k}(F) \cap F_{s_i} = \gamma_k(F') \cap F_{s_i} \text{ if } h_i \leqslant k,$$

and

$$\gamma_k(F') \leq \gamma_{h_i}(F') \text{ if } h_i \leqslant k$$

and

$$\gamma_{h_i}(F') \cap \gamma_{s_i}(F) \cap F_{s_i} = \gamma_{h_i}(F') \cap F_{s_i}$$

$i = 1, \ldots, r$, we obtain

$$I_{(m)} \cap F_{s_i} = (\gamma_{h_i}(F') \cap F_{s_i}) + (\delta_3(F) \cap F_{s_i}),$$

$i = 1, \ldots, r$. But

$$I_{(m)} \cap F_{s_i} = (\gamma_{h_i}(F') + \delta_3(F)) \cap F_{s_i} \tag{2}$$

$i = 1, \ldots, r$. Similarly,

$$I_{(m)} \cap F_{s_i} = (\gamma_{h+1}(F') + \delta_3(F)) \cap F_{s_i} \tag{3}$$

Since $I_{(m)}, I_{(m+1)}$ are fully invariant ideals of F, we have

$$I_{(m)}/I_{(m+1)} = \bigoplus_{i \geq 1} ((I_{(m)} \cap F_i) + I_{(m+1)})/I_{(m+1)} \tag{4}$$

But $((I_{(m)} \cap F_{s_i}) + I_{(m+1)})/I_{(m+1)}$ is isomorphic as KG-module to

$$(I_{(m)} \cap F_{s_i})/(I_{(m)} \cap F_{s_i} \cap I_{(m+1)}) = (I_{(m)} \cap F_{s_i})/(I_{(m+1)} \cap F_{s_i}) \tag{5}$$

From (1), (2), (3), (4) and (5),

$$D^t \otimes U \geqslant I_{(m)}/I_{(m+1)}.$$

This completes the proof of the Theorem.

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