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COHOMOLOGY THEORY ON NON COMMUTATIVE ALGEBRAS OF CONTINUOUS FUNCTIONS

BY

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Abstract. In this note, we generalize results of H. Kamowitz ([20]) and B.E. Johnson ([15]), from the commutative case to non commutative.

1. Introduction

The cohomology theory of associative linear algebras over an arbitrary field was initiated and developed by Hochschild [11]. In [23] J. Ringrose is concerned with cohomology groups of operator algebras and especially with von Neumann algebras. A non Neumann algebra $\mathfrak{R}$ has no outer derivations ([25] and [6]), this result can easily reformulated, in terms of the Hochschild cohomology theory, as the assertion that $H^1(\mathfrak{R},\mathfrak{R})=0$. For type I von Neumann algebra is proved that $H^n(\mathfrak{R},\mathfrak{R})=0$ $(n=1,2,...)$. Cohomology theory for operator algebras (in particular for von Neumann algebras) was initiated also in a series of three papers [23], [15], [16].

Also H. Kamowitz [20] in 1962 extended the cohomology theory of Hoschschild to commutative Banach algebras. In particular proved $H^2(C(X), C(X))=0$ where X is compact Hausdorff space. Also B.E. Johnson ([15], [16]) proved analogous theorems for commutative Banach algebras. For example proved $H^n(C(X),C(X))=0$ $(n=1,2,...)$ if X is a compact metrisable space.

1.1. Suppose that E is a complex Banach algebra and X is a two-sided E-module. We say that X is **a Banach E-module** if X is a Banach space and the bilinear mappings

$$\begin{matrix}(a,x)\to ax\\(a,x)\to xa\end{matrix}\quad ExX\to X$$

are bounded.

Given a complex Banach algebra E and a Banach E-module X, for each positive integer n we denote by $L_c^n(E,X)$ the (normed) linear space which consists of all bounded n-linear mappings from $E^n(=ExEx...xE)$ into X. We also define $L_c^0(E,X) \equiv X$. When $n \geqslant 0$ we refer to elements of $L^n(E,X)$ as n-cochains.

For each positive integer n, we define the **coboundary operator** $\Delta: L^{n-1}(E,X) \to L^n(E,X)$ as follows

If $f \in L_c^n(E,X)$, Δf in $L_c^{n+1}(E,X)$ is defined by

$$\Delta f(a_1,a_2,...,a_{n+1}) = a_1 f(a_2,...,a_{n+1}) + \sum (-1)^i f(a_1,...,a_i,a_{i+1},...,a_{n+1}) +$$
$$+(-1)^{n+1} f(a_1,...,a_n)a_{n+1} \quad (a_i,...,a_n \ E)$$

For $n=1,2,...$ we introduce two linear subspaces of $L_c^n(E,X)$. The range of $\Delta: L^{n-1}(E,X) \to L^n(E,X)$

is denoted by $B_c^n(E,X)$ and its elements are called **n-coboundaries** or **n-cohains.**

The kernel of $\Delta: L_c^n(E,X) \to L_c^{n+1}(E,X)$ is denoted by $Z_c^n(E,X)$ and its elements are called **n-cocycles**. Since $\Delta^2 = 0$ ([11]) it follows that $B^n(E,X) \subseteq Z^n(E,X)$ and (ignoring topologies on these spaces) we can form the quotient linear space

$$H_c^n(E,X) = Z_c^n(E,X)/B_c^n(E,X)$$

We call $H_c^n(E,X)$ the **n-dimensional** (continuous) cohomology groups of E with coefficients of X (see also: [13]).

1.2. A **derivation** δ on an algebra E is a linear transformation on the algebra such that

$$\delta(xy) = \delta(x) \cdot y + x\delta(y)$$

A derivation on E is said to be bounded if $\sup \|\delta(x)\| = \|\delta\| < +\infty$. An element α of some possible larger algebra is said to implement δ if

$$\delta(x) = \alpha x - x\alpha \quad (x \in E)$$

and δ is said to be **inner** if such an α can be found in E. Otherwise δ is said to be outer [6].

1.3. $B^1_c(E,X) = B^1(E,X) =$ the set of all inner derivations from E into X.

$Z^1_c(E,X) =$ the set of all continuous derivations from E into X.

$Z^1(E,X) =$ the set of all derivations from E into X.

$H^1_c(E,X) = \{0\}$ iff all continuous derivations from E into X are inner.

$H^2(E,E) = 0 \Leftrightarrow$ T billinear map

$$T: E \times E \to E: xT(y,z) - T(xy,z) + T(x,yz) - T(x,y)z = 0$$

there is a linear map $S: E \to E$ such that $T(x,y) = xS(y) - S(xy) + S(x)y$

1.4. (i) Let P, Q, R be normed linear spaces over the field K (complex or real numbers). Denote the space of linear K-valued functionals f defined on P and normed in this manner

$$\| f\| = \sup | x(x) |$$

as usual by P^*.

Let $f: P \times Q \overset{\text{into}}{} R$ be a bounded bilinear operation.
We define:

$f^*: R^* \times P \to Q^*: (f^*(z^*,x))(y) = z^*(f(x,y))$

$f^{**}: Q^{**} \times R^* \to P^*: f^{**}(y^{**},z^*)(x) = y^{**}(f^*(z^*,x))$

$f^{***}: P^{**} \times Q^{**} \to R^{**}: f^{***}(x^{**},y^{**})(z^*) = x^{**}(f^{**}(y^{**},z^*))$

(see [1]). Then, we have $f^{***}(x,y) = f(x,y)$

(ii) A bounded bilinear map f is called **regular** iff

$$\bar{y} \to f^{***}(\bar{x},\bar{y})$$

is w^*-continuous for fixed $\bar{x} \in P^{**}$.

(iii) Let E be a t.v.s. A net (x_α) in E is said to be **weakly convergent** if there is an $x \in E$: $x^*(x) = \lim x^*(x_\alpha)$ $\forall\, x^* \in E^*$. The point x is called a **weak limit** of the net, and the net (x_α) is said to converge weakly on x.

(iv) Every sequence (x_n) such that $(x^* x_n)_{n \in N}$ is a Gauchy sequence of scalars for each $x^* \in E^*$, is called a weak Gauchy sequence ([7, 11, 3.23, p. 67]).

Main results

2. In order to prove which is stated in the abstract we prove the following:

Proposition 2.1. Let X be any compact Hausdorff space. Let $x_k, y_k \in C(X,E)$ $1 \leqslant k \leqslant n$ where E is a unital C*-algebra. We define elements x,y in C(X,E) by setting

$$x(t) = \sum_{k=1}^{n} \|x_k(t)\| I$$

$$y(t) = \sum_{k=1}^{n} \|y_k(t)\| I$$

where I, is the identity of the algebra E.
Then,

$$\| \sum_{k=1}^{n} x_k \otimes y_k \| \leqslant \| x \| \ \| y \|$$

in the space

$$C(X,E) \otimes C(X,E), \ ([21, \text{ p. } 159]).$$

Proof: In $C(X,E) \otimes C(X,E)$ we consider an element of the form

$$u = \sum_{i=1}^{n} \frac{1}{n} (x_1 + \varepsilon^k x_2 + \cdots + \varepsilon^{k(n-1)} x_n) \otimes (y_1 + \varepsilon^{-k} y_2 + \cdots + \varepsilon^{-k(n-1)} y_n)$$

where ε is a primitive n-th root of unity. Using the bilinearity of $\otimes$ and combining similar terms we obviously obtain the equality

$$u = \sum_{i,j=1}^{n} \lambda_{ij} x_i \otimes y_j$$

where $\lambda_{i,j} = \frac{1}{n} \sum_{k=1}^{n} \varepsilon^{k(i-1)} \varepsilon^{-k(j-1)} = \frac{1}{n} \sum_{k=1}^{n} \varepsilon^{k(i-j)}$

If $i \neq j$ $\lambda_{ij} = 0$

If $i = j$ $\lambda_{ij} = 1$ and $u = \sum_{k=1}^{n} x_k \otimes y_k$

$$\| \sum_{k=1}^{n} x_k \otimes y_k \| = \| u \| \leqslant \sum_{k=1}^{n} \| \frac{1}{n} (x_1 + \varepsilon^{-k} x_2 + \cdots + \varepsilon^{k(n-1)} y_n \| \cdot$$

$$\cdot \| y + \varepsilon^{-k} y_2 + \cdots + \varepsilon^{k(n-1)} y_n \|$$

Now,

$$\| x_1 + \varepsilon^k x_2 + \cdots + \varepsilon^{k(n-1)} x_n \| \leqslant \| x \|$$

$$\| y_1 + \varepsilon^{-k} y_2 + \cdots + \varepsilon^{-k(n-1)} x_n \| \leqslant \| y \|$$

and thus

$$\| \sum_{k=1}^{n} x_k \otimes y_k \| \leqslant \| x \| \cdot \| y \|$$

in $C(X,E) \otimes C(X,E)$ (See also: [10]).

2.2. Let X be a compact metrisable space and E a unital C^*-algebra with identity I. Let d be a metric determining the topology on X.
For each $k \in Z$ define

$$f_k \colon [0, +\infty) \to [0,1] \text{ by}$$

$f_k(x) = 0$ if $0 \leqslant x \leqslant 2^{-k-1}$ or $2^{-k-1} \leqslant x$

$f_k(2^{-k}) = 1$

f_k is linear on $[2^{-k-1}, 2^{-k}]$ and on $[2^{-k}, 2^{-k+1}]$

Fix $\omega \in X$ and let

$$M_\omega = \{ a \in C(X,E) \colon a(\omega) = 0 \} \subset C(X,E)$$

the two-sided closed ideal.
For each $k \in Z$ define $\tilde{g}_{\omega,k} \in C(X)$ by $g_{\omega,k}(t) = f_k(d(\omega,t))^{1/2} \quad (t \in X)$
Define

$$g_{\omega,k} \in C(X,E) \text{ by } g_{\omega,k}(t) = \tilde{g}_{\omega,k}(t) I \quad (t \in X)$$

Proposition 2.3. Let $a \in M_\omega$, then $\sum g_{\omega,k} \otimes g_{\omega,k} a$ converges in $C(X,E) \otimes C(X,E)$ (to an element of $M_\omega \otimes M_\omega$).

Proof: Since X is compact there exists $N_o \in \mathbb{N}$ such that $\sup d(t,t') < 2^{N_o}$ for all t, t' in X, so (for any ω in X) $g_{\omega,k} = 0$ for $k < -N_o$. Then we have only to consider $\sum_{K=-N_o}^{\infty} g_{\omega,k} \otimes g_{\omega,k} a$ (whatever point ω is in X). Let $\varepsilon > 0$. Since $a \in M_\omega$ there exists $N_1 \in \mathbb{N}$ such that $\| a(t) \| < \varepsilon$ whenever $d(\omega,t) < 2^{-N_1}$.

Let $m,n>N_1$, $m<n$. Then

$$\sum_{k=m}^{n} \| g_{\omega,k}(t)\alpha(t)\| \leqslant \varepsilon \sum_{k=m}^{n} \| g_{\omega,k}(t)\|$$

for all t in X, since, for all $k>N_1$ we have $g_{\omega,k}(t)=0$ whenever $d(\omega,t)\geqslant 2^{-N_1}$.

Further, $\sum_{k=-\infty}^{\infty} \tilde{g}_{\omega,k}(t)\leqslant\sqrt{2}$ for all $t\in X$, because, for each $x\in[0,+\infty)$ $f_k(x)=0$ for all but at most two values of k, and $\sum_{k=-\infty} \sqrt{f_k(x)}$ is clearly $\leqslant 2\cdot\frac{1}{\sqrt{2}}=\sqrt{2}$ ([10 p. 405]).

It follows from 2.1 that

$$\| \sum_{k=m}^{n} g_{\omega,k}\otimes g_{\omega,k}\alpha\| \leqslant \sup_{t\in x}\| \sum_{k=m}^{n} \| g_{\omega,k}(t)I\| \cdot \sup\| \sum_{k=m}^{n} \| g_{\omega,k}(t)\alpha(t)\| I\| =$$

$$= \sup\Big(\sum_{k=m}^{n} \| g_{\omega,k}(t)\|\Big) \sup\Big(\sum_{k=m}^{n} \| g_{\omega,k}(t)\alpha(t)\|\Big) \leqslant \varepsilon(\sup \sum_{k=m}^{n} \| g_{\omega,k}(t)\|)^2 =$$

$$= \varepsilon(\sup \sum_{k=m}^{n} \tilde{g}_{\omega,k}(t))^2 \leqslant 2\varepsilon$$

Thus, the sequence of partial sums of $\sum_{K=-N_0}^{\infty} g_{\omega,k}\otimes g_{\omega,k}\alpha$ is Gauchy and so the series converge in $C(X,E)\otimes C(X,E)$ (to an element of $M_\omega\otimes M_\omega$ since $g_{\omega,k}\in M_\omega$).

Proposition 2.4. The map $J_\omega:\alpha\to \sum_{k=-\infty}^{\infty} g_{\omega,k}\otimes g_{\omega,k}$: $M_\omega\to M_\omega\otimes M_\omega$ is a bounded linear map, with $\|J_\omega\|\leqslant 2$.

Proof: J_ω is clearly linear. Exactly as in the proof of 2.3 we also have for each n

$$\| \sum_{k=-\infty}^{n} g_{\omega,k}\otimes g_{\omega,k}\alpha\| = \| \sum_{k=-N_0}^{n} g_{\omega,k}\otimes g_{\omega,k}\alpha\| \leqslant \sup_{t\in x}\| \sum_{k=-N_0}^{n} \| g_{\omega,k}(t)\| I\| \cdot$$

$$\cdot \sup\| \sum_{k=-N_0}^{n} \| g_{\omega,k}(t)\alpha(t)\| I\| \leqslant \sup_{t\in x}\|\alpha(t)\| \cdot \Big(\sup\Big(\sum_{k=-N_0}^{n} \| g_{\omega,k}(t)\| \Big)^2 =$$

$$= \|\alpha\| \Big(\sup \sum_{k=-N_0}^{n} \tilde{g}_{\omega,k}(t) \Big)^2 \leqslant 2\|\alpha\|$$

$\|J_\omega(\alpha)\| = \| \sum_{k=N_0}^{\infty} g_{\omega,k} \boxtimes g_{\omega,k}\alpha \| \leqslant 2\|\alpha\|$ for all $\alpha \in M_\omega$, and hence $\|J_\omega\| \leqslant 2$.

2.5. Now, since $T \in L^2(C(X,E),C(X,E))$ we define $S \in L^2(C(X,E),C(X,E))$ as follows: for α in $C(X,E)$ and ω in X define

$$(S\alpha)(\omega) = -T(J_\omega(\alpha - \alpha(\omega)1)(\omega) + \alpha(\omega)T(1,1)(\omega)$$

where 1 denotes the element of $C(X,E)$ sending every element of X to I (It is obvious that $J_\omega(\alpha - \alpha(\omega)1) \in C(X,E) \boxtimes C(X,E)$ and by $T(J_\omega(\alpha - \alpha(\omega)1)$ is meant the image of $J_\omega(\alpha - \alpha(\omega)1)$ under the linear map: $C(X,E) \boxtimes C(X,E) \to C(X,E)$ induced by T).

We have to check that S really does map $C(X,E)$ linearly into $C(X,E)$.

Firstly, linearity is clear. Secondly, S maps $C(X,E)$ into $B(X,E)$, the space of bounded functions from X to E, because for all t in X we have

$$\|(S\alpha)(t)\| = \|-T(J_t(\alpha - \alpha(t)1)(t) + \alpha(t)T(1,1)(t)\| \leqslant$$

$$\leqslant \sup_{\omega \in X} \|-T(J_\omega(\alpha - \alpha(\omega)1)(\omega) + \alpha(\omega)T(1,1)(\omega)\|$$

If $\|\alpha\| \leqslant 1$, $\|\alpha - \alpha(\omega)\| \leqslant 2$ and by previous Lemma

$$\|S(\alpha)(t)\| \leqslant \|T\| \cdot 2 \cdot 2 + \|T\| = 5\|T\|$$

Thus, $\sup_{t \in X} \|(S\alpha)(t)\| = \|S\| \leqslant 5\|T\|$

If $\|\alpha\| \leqslant 1$, $\|S\| \leqslant 5\|T\|$.

In particular $S\alpha \in B(X,E)$ for all $\alpha \in C(X,E)$. It remains to prove that $S\alpha$ $C(X,E)$ for all α in $C(X,E)$.

As $(\alpha_1,\alpha_2,\omega) \to T(\alpha_1,\alpha_2)(\omega): (C(X,E))^2 \times X \to \mathbb{C}$ is continuous, it is enough to show that for each α in $C(X,E)$ the map $\omega \to J_\omega(\alpha - \alpha(\omega)1): X \to C(X,E) \boxtimes C(X,E)$ is continuous.

Since J_ω is uniformly bounded in ω ($\|J_\omega\| \leqslant 2$ for every ω) it follows, using familiar facts about uniform convergence, that we need only show the continuity of $\omega \to J_\omega(\alpha - \alpha(\omega)1)$ at each point of X for all α in a dense subset of $C(X,E)$. Fix ω in X and consider the dense subset of $C(X,E)$ consisting of those elements of $C(X,E)$ which are constant in a neighborhood of α, suppose that α is constant in the neighborhood $\{t: d(\omega,t) \leqslant 2^{-m}\}$ of ω. Then, for any ω' with $d(\omega,\omega') < 2^{-m-1}$ we have that $g_{\omega',k}(\alpha - \alpha(\omega')1) = 0$ whenever $k > m+2$ (because $g_{\omega',k}(t) = 0$ for all t with $d(t,\omega') > 2^{1-k}$ and if $k > m+2$ this includes all t with $d(t,\omega') > 2^{-m-1}$ while $(\alpha - \alpha(\omega')1)(t) =$

$=(\alpha(t)-\alpha(\omega'))1=0$ for all t with $d(t,\omega')\leqslant 2^{-m-1}$. Since all such points t satisfy $d(\omega,t)\leqslant 2^{-m}$ so that $\alpha(t)=\alpha(\omega')$). It follows clearly that, if $\omega'\to\omega$ $g_{\omega',k}\to g_{\omega,k}$ $\forall\ k\in N$.

If $\omega'\to\omega$ $d(x,\omega')\to d(x,\omega)$ (since d is (uniformly) continuous on X).

$$g_{\omega',k}(x)=(f_k\circ d(x,\omega'))^{1/2}\to(f_k\circ d(x,\omega))^{1/2}=g_{\omega,k}(x)$$

The convergence is uniform.

$$J_{\omega'}(\alpha-\alpha(\omega')1)=\sum_{k=-\infty}^{\infty} g_{\omega',k}\otimes g_{\omega',k}(\alpha-\alpha(\omega')1)=\sum_{k=-N_0}^{m+2} g_{\omega',k}\otimes g_{\omega',k}(\alpha-\alpha(\omega')1)$$

$$\to\sum_{k=-N_0}^{m+2} g_{\omega,k}\otimes g_{\omega,k}(\alpha-\alpha(\omega)1)=J_\omega(\alpha-\alpha(\omega)1)$$

as $\omega'\to\omega$ so the map

$$\omega\to J_\omega(\alpha-\alpha(\omega)1)$$

is continuous at ω. This concludes that the proof that $S\alpha\in C(X,E)$ for all $\alpha\in C(X,E)$.

2.6. If $T\in L^n(C(X,E),C(X,E))$ we define $S\in L^{n-1}(C(X,E),C(X,E))$ by

$$S(\alpha_1,\ldots,\alpha_{n-1})(\omega)=-T(J_\omega(\alpha_1-\alpha_1(\omega)1,\alpha_2,\ldots,\alpha_{n-1}-\alpha_{n-1}(\omega)1)(\omega)+$$

$$+\alpha_1(\omega)T(1,1,\alpha_2,\ldots,\alpha_{n-1})(\omega)+(-1)^n\alpha_{n-1}(\omega)T(\alpha_1-\alpha_1(\omega)1,\alpha_2,\ldots,\alpha_{n-2},1,1)(\omega)$$

and then, for all $t\in X$

$\|S(\alpha_1,\ldots,\alpha_{n-1})(t)\|\leqslant\|T\|\cdot 2\cdot 2\cdot 2+\|T\|+\|T\|\cdot 2=11\|T\|$ as in [15, p. 447].

3. In this section we generalize theorem of H. Kamowitz as we indicate to abstract. H. Kamowitz [20] proved that X is a compact Hausdorff space and R is a banach space, then every bounded bilinear map f of $C(X)\times C(X)$ into R is regular. We prove the same result for the algebra $C(X,M_n)$.

Proposition 3.1. Let E,F be C^*-algebras and β'' C^*-norm on the algebraic tensor product $E^{**}\otimes F^{**}$. Then

$$E^{**}\underset{\beta''}{\otimes}F^{**}\subset(E\underset{\beta}{\otimes}F)^{**}$$

Proof: We choose to think of E and F as acting in its universal representation over Hilbert spaces H_1,H_2. Thus E^{**} can be identified with $\overline{E}$, the

closure of E with respect to the weak-operator topology ([5, 2.7.6, 12.1]).

We have $E\underset{\beta}{\otimes}F \subseteq E^{**}\underset{\beta''}{\otimes}F^{**}$, $\beta=\beta''/E\otimes F$ ([2]). By the definition of an arbitrary element of $E^{**}\otimes F^{**}$ we have the result.

The space $E^{**}\underset{\beta''}{\otimes}F^{**}$ never coincides with $(E\underset{\beta}{\otimes}F)^{**}$ ([24]).

Proposition 3.2. Let X be a compact Hausdorff space, then $C(X,M_n)$ is a weakly complete space (i.e. every weak Cauchy sequence has a weak limit).

Proof: $C(X,M_n)^* = C(X)^*\otimes M_n$ (within an isomorphism, β arbitrary C^*-norm [21]). It is known ([20] and [7, 11, 3.25 *p. 69, Part I*]) that $C(X)^*$ and M_n are weakly complete spaces. We apply the definition of weak Cauchy sequences to the above spaces and the result is obvious from the definition of an arbitrary element of tensor product $C(X)^*\underset{\beta}{\otimes}M_n$.

3.3. Let $T: E \to F$ be bounded linear operator and S closed unit ball in E. The operator T is said to be weakly compact if the weak closure of T(S) is compact within the topology of F.

If T_1, T_2 are weakly compact $T_1\otimes T_2$ is weakly compact ([7, p. 482-484]).

Proposition 3.4. The bounded linear operator

$$T: C(X,M_n) \to C(X,M_n)^*$$

is weakly compact.

Proof: Let $T: C(X)\underset{\alpha}{\otimes}M_n \to C(X)^*\underset{\alpha}{\otimes}M_n$ be a linear operator. Every

$T \in L(C(X)\otimes M_n, C(X)^*\otimes M_n)$ can be written as $T=\sum_{i=1}^{n} T_{1i}\otimes T_{2i}$ where $T_{1i}: C(X)\to C(X)^*$, $T_{2i}: M_n \to M_n$ since ([4, I, p. 206, 274]),

$$L(C(X)\otimes M_n,\ C(X)^*\otimes M_n) \simeq L(C(X),\ C(X)^*)\otimes L(M_n,M_n)$$

T_{1i}, T_{2i} are weakly compact operators and therefore their linear combination is a weakly compact operator.

3.5. Let E be a normed linear space and E^{**} the conjugate of the Banach space E^*. The map $k: x \to \hat{x}: E \xrightarrow{\text{into}} E^{**}$, $\hat{x}(x^*)=x^*(x)$ is called the natural embedding of E into E^{**} ([7, 11, 18, Part 1]).

Proposition 3.6. If X is a compact Hausdorff space and R is a Banach space, then every bounded bilinear map f of $C(X,M_n)\times C(X,M_n)$ into R is regular.

Proof: By 1.4 (iii) and the remark that $C(X,M_n)^*$ is weakly complete, the proof is mutatis mutandis of [20, Th. 4.2].

Proposition 3.7. Let X compact Hausdorff space. If f is a bounded $(2,C(X,M_n))$-cocycle of $C(X,M_n)$, then f^{***} is a bounded $(2,C(X,M_n)^{**})$-cocycle of $C(X,M_n)^{**}$.

Proof: Suppose $\bar{x},\bar{y},\bar{z} \in C(X,M_n)^{**}$ and $\{x_k\}_{k\in K}$, $\{y_n\}_{n\in N}$, $\{z_p\}_{p\in P}$ nets of $C(X,M_n)$ which approach x,y,z respectively, in the w*-topology of $C(X,M_n)^{**}$. Since f is $(2,C(X,M_n))$-cocycle, we have

$$x_k f(y_n,z_p) - f(x_k y_n,z_p) + f(x_k,y_n z_p) - f(x_k,y_n)z_p = 0$$

we have

$$\bar{x}f^{***}(\bar{y},\bar{z}) - f^{***}(\bar{x}\bar{y},\bar{z}) + f^{***}(\bar{x},\bar{y}\bar{z}) - f^{***}(\bar{x},\bar{y})\bar{z} = 0$$

by [20, Th. 4.3, Th. 4,4] and Proposition 3.6.

Proposition 3.8. Let f be a bilinear map of an algebra E into E. If g is a linear map of E into E and $\Delta g = f$, then for $2 \leqslant n \in N$, $u \in E$

$$\sum_{i=0}^{n-1} u^{n-i-1} g(u) u^i - g(u^u) = \sum_{i=0}^{n-2} u^i f(u, u^{n-i-1})$$

Proof: $f(u,u) = \Delta g(u,u) = ug(u) - g(u^2) + g(u)u$ (1)

$f(u,u^2) = \Delta g(u,u^2) = ug(u^2) - g(u^3) + g(u)u^2$ (2)

We multiply (1) by u and add with (2)

$f(u,u^2) + uf(u,u) = u^2 g(u) + ug(u)u + g(u)u^2 - g(u^3)$ (3)

$\Delta g(u,u^3) = f(u,u^3)$

$\Delta g(u,u^3) = ug(u^3) - g(u^4) + g(u)u^3 = f(u,u^3)$ (4)

We multiply (3) by u and add with (4)

$f(u,u^3) + uf(u,u^2) + u^2 f(u,u) = u^3 g(u) + u^2 g(u)u + ug(u)u^2 - g(u^4)$

and so on inductively we have the result

$$u^{n-1}g(u) + u^{n-2}g(u)u + u^{n-3}g(u)u^2 + \cdots + ug(u)u^{n-2} + g(u)u^{n-1} - g(u^n) =$$

$$= f(u,u^{n-1}) + uf(u,u^{n-2}) + \cdots + u^{n-2}f(u,u)$$

Proposition 3.9. Suppose f is a bilinear map of an algebra E into a commutative E-module X. If g is a linear map of E into X and $\Delta g = f$, then for $2 \leqslant n \in \mathbf{N}$ and $u \in E$ we have

$$nu^{n-1}g(u) = \sum_{i=0}^{n-2} u^i f(u, u^{n-i-1})$$

Proof: Obvious by the Proposition 3.8 and [20, lemma 4.5, p. 365].

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