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DETERMINATION OF ALL RIGID GROUPS OF ORDER NOT EXCEEDING 60

BY

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It has been shown by Heineken and Liebeck [3], that for every finite group K there is a p-group P such that $\mathrm{Aut}(P)/\mathrm{Aut}_c(P) \simeq K$, where $\mathrm{Aut}_c(P)$ is the group of all central automorphisms of P. The proof used the construction of a digraph D(K) depending on a set of generators of K, which in turn led to the p-group P such that (under certain conditions on D(K)) the group K could be viewed as a semiregular permutation group of a suitable basis of P.

In this paper we want to decide for which groups of order not exceeding 60 we can deduce the existance of a P such that K operates regularly on a basis of P; these groups we will call rigid.

For this, we construct a digraph $D_M(K)$ depending on the group K and a generating set M of K such that arrows correspond to generators belonging to M. More explicitly, we call K rigid, if

(I) $\mathrm{Aut}(D_M(K)) \simeq K$ *for some generating set* M of K, and furthermore, for the transition to P, we need (see Lemma 2.1.1 [4])

(II) The digraph $D_M(K)$ does not possess closed paths of length smaller than 5.

We begin with some results established earlier ([4] Lemma 3.1.1) and needed later on.

Lemma 1.1.1. Let G be a finite group generated by two elements x,y and assume $o(x)=5$ and $o(y)>5$. The graph constructed by G using the generators x and y can be used for the construction of a corresponding nilpotent group, if we have:

(i) $[y^2,x] \neq 1$, (ii) $(xy)^2 \neq 1$, (iii) $(xy^{-1})^2 \neq 1$

Lemma 1.1.2. Assume that G is finite and is generated by two elements of order 5 e.g. x and y such that $(xy)^2 \neq 1$ $(xy^{-1})^2 \neq 1$ and G is non-abelian. The graph constructed by x and y is suitable, if and only if there is no automorphism of G interchanging x and y.

In the case of non-rigidity, two basic Lemmas are stated with proof:

Lemma 1.1.3. If G possesses a normal subgroup N of exponent 2 and G/N is isomorphic to a dihedral group, then G has no rigid description.

Proof: In any generating set of G/N there is an element xN of order 2 since $G/N \simeq D_n$. So in any generating set of G there is an element x with x^2 N but N is of exponent 2 so $x^4=1$.

Lemma 1.1.4. If $G \simeq D_m \times D_n$, then G, has no rigid two generator description.

Proof: Assume $G \simeq A \times B$, $A \simeq D_m$, $B \simeq D_n$. Then $G = <x,y>$ where $x = a_1 b_1$, $y = a_2 b_2$, $a_i \in A$, $b_i \in B$, $i=1,2$.
Since $<a_1, a_2> = A$ one of the two elements has order 2, say a_1. Since $a_1 b_1$ is not of order 2 and $<b_1, b_2> = B$, b_1 is not of order 2 but b_2 is. Also a_2 is not of order 2, but then $(a_1 a_2)^2 = (b_1 b_2)^2 = 1$ and so $(xy)^2 = 1$.

In this context it may be mentioned that direct products of three dihedral groups may well be rigid.

All non abelian groups of order 2p, o a prime have no suitable graph, in fact all dihedral groups and all (generalized) quaternion groups, also all non-abelian groups of order 3p and 4p, p a prime are not rigid.

We will generally use the notation of [2] and the tables of [1]. All groups considered in this paper are finite. The digraph $D_M(K)$ is described in [3] as laid down in [4]. We need few generators for the group K and the number of generators of G is $|K|$.

1.2. Obstacles of a group against rigidness

The graph is suitable if and only if there are no circuits of length 4 or less in the graph. Every point in the graph has the same properties with respect to arrows, paths and so on. If we have no circuits of length smaller than 5 in the graph, the number of points is bounded from below depending on the number of generators. We deduce at once that all paths of length 0, 1, 2, beginning at a fixed point P, lead to different end points.

First we consider the path of length 0, which is just this point, its number is 1. Second we consider the open paths of length 1, which is an arrow beginning or ending in the given point. In order to have all paths of length 1 we must have 2k points. Third we consider the open paths of length 2, we must have

$2k(2k-1)$ points since every point of $2k$ points must connected with $2k-1$ points (excluding the sequences $x.x^{-1}$).
So if the number of points is less than

$$1+2k+2k(2k-1)=(2k)^2+1$$

there is at least one circuit of length smaller than 5.

For instance, a description of any group K with $|K| \leqslant 60$ by more than 3 generators is unsuitable since $8^2+1=65>60$.

1.3. The group A_5

Lemma 1.3.1. Assume that a and b are two elements of order 5 in A_5 such that $<a,b>=A_5$. Then there is an automorphism of A_5 interchanging a and b.

Proof: We may consider $A_5 \leqslant Aut(S_5)$ since A_5 has trivial center. The automorphisms are then induced by conjugation. We have proved the Lemma if we can show that for every element x of order 5 with $<a,x>=A_5$ there is an element of order 2, say y, such that $y^{-1}ay=x$ ($y^{-1}xy=a$, $y^2=1$). This again is equivalent to the following statement: Every coset of $C_{A_5}(a) = \; = <a>$ which is not contained in the Normalizer $N_{A_5}<a>$ contains an element of order 2. We want to prove this statement.

Assume that a certain coset $<a>t$ contains two elements g, h of order 2. Then $g=a^kt$, $h=a^lt$ and $gh^{-1}=a^ktt^{-1}a^{-l}=a^{k-l}$, is a power of a and $<g,h>=<g,a>$ is a dihedral group contained in the $N<a>$ since $<a>$ is of order 5 and is of index 2 in $<g,a>$ so $<a> <| <g,a>$ and $g \in N<a>$. Thus $<g,a> \;\; N<a>$ in fact all elements in $<a>t$ have order 2. We derive

(a) There is exactly one coset of $<a>$ in $N<a>$ which contains several elements of order two, and it contains five of them.

(b) The remaining 20 cosets outside $N<a>$ contain at most one element of order two. To show that the number of cosets outside are 20 we find the number of elements of order 2 in S_5 is 25.

We have 10 transpositions and 15 elements of the form $(r,s)(v,w)$ with $r \neq s \neq v \neq w$ so the total number of elements of order 2 is 25.

Using the statements (a) and (b) we see that all of the cosets outside $N<a>$ contain one element of order two so there exists an automorphism interchanging the two elements of order 5.

Corollary 1.3.2. A_5 has no-rigid representation by two generators.

1.4. The group A_5 can not be generated "rigidly" by two elements but the group A_5 might be generated by 3 elements of order 5. We consider this possibility more closely here. All elements of order 5 belong to two conjugate classes of 5-cycles containing 12 elements each:

$C_1 = \{(12345), (13542), (14352), (15432), (14523), (15243),$
$(13254), (15324), (14235), (13425), (12453), (12534)\}$
$C_2 = \{(14325), (15234), (15423), (13524), (14532), 14253),$
$(13452), (12435), (13245), (12354), (12543), (15342)\}$

We have the cases:
Case a: Two elements are in different classes. In this case there is a product of two elements xy or xy^{-1} of order 2. It is sufficient to check just one element since if we have chosen two elements x and y then everything we say about x and y (with respect to relations) is also true for x and $x^{-i}yx^{i}$. If there are 4-paths with respect to x and y^{-1}, x and $x^{-i}yx^{i}$ and with respect to x and $x^{-i}y^{-i}x^{i}$. These elements $y, y^{-1}, x^{-i}yx^{i}, x^{-i}y^{-i}x^{i}$ and some powers of x the whole conjugacy class of y.

Case b: All three generators are in the same conjugate class. In this case we may take $x = (12345)$ and $y = (12453)$. To avoid 3-paths we are not allowed to take z to be equal to xy or yx: $z \neq xy = (14352)$, $z \neq (yx) = (13254)$ and to avoid 4-paths z should be different from $x^{-1}yx$ and xyx^{-1}; $z \neq x^{-1}yx = (14235)$, $z \neq (xyx^{-1}) = (13425)$ and their inverses.

By this all elements of the conjugacy class containing x and y are excluded for z. So there is no three-generator rigid presentation, and in fact no rigid presentation at all, of A_5.

List of all rigid groups of order not exceeding 60, using two generators.

$\|K\|$	rigid isomorphism class
24	$C_3] C_8$
27	$C_9] C_3$
30	$C_3] C_{10}$, $C_5] C_6$
32	$C_{16}] C_2 = \langle a, b \mid a^{16} = 1,\ b^2 = 1,\ bab = a^9 \rangle$
36	$C_3 \times (C_3] C_4)$
40	$C_5] C_8$ (two cases possible), $C_5 \times Q_8$, $C_5 \times D_4$

42	$C_7]C_6$ (three cases possible), $D_3 \times C_7$
48	$C_3]C_{16}$ $C_8 \times (C_3]C_2)$, $C_2 \times (C_3]C_8)$, $GL(2,3)$
	$C_8]C_6$ (two cases possible)
	$\langle x,y \mid x^8 = y^{12} = 1,\ x^4 = y^6,\ x^{-1}yx = y^5 \rangle$
	$\langle x,y \mid x^6 = y^{12} = 1,\ y^4 = x^4,\ xyx^{-1} = y^7 \rangle$
	$\langle x,y \mid x^{24} = y^{12} = 1,\ x^{-4} = y^2,\ y^{-1}xy = x^7 \rangle$
	$\langle x,y \mid x^{12} = y^{12} = 1,\ x^4 = y^4,\ y^{-1}xy = x^7 \rangle$
50	$C_5 \times D_5$
54	$C_9]C_6$ (2), $C_9 \times C_3]C_2$
55	$C_{11}]C_5$
56	$C_7]C_8$, $(C_2 \times C_2 \times C_2)]C_7$, $C_7]D_4$, $Q_8 \times C_7$
60	$C_5 \times A_4$, $C_5 \times D_6$, $C_{15}]C_4$
	$C_3 \times T_1 = \langle x,y \mid x^{30} = y^{20} = 1,\ x^3 = y^{-2},\ y^{-1}xy = x^{11} \rangle$
	$C_3 \times T_2 = \langle x,y \mid x^{12} = y^5 = 1,\ x^{-1}yx = y^2 \rangle$

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