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FIBONACCI NUMBERS IN PERMUTATIONS

BY

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Abstract : *It is shown that the cardinality of three different sets of permutations is equal to the numbers of Fibonacci.*

1. Introduction

If S_n is the set of all permutations of $[n] = \{1,2,...,n\}$ then each element of S_n is denoted by $\sigma = \sigma(1)\sigma(2)...\sigma(n)$.

A pattern or up-down sequence of a permutation σ is the "word" $w=z_1z_2...z_{n-1}$ of n-1 characters with

$$z_i = \begin{cases} + , \text{ if } \sigma(i) < \sigma(i+1) \\ \\ - , \text{ if } \sigma(i+1) < \sigma(i) \end{cases}$$

For example the pattern of the permutation σ = 67135428 is w=+-++--+.

On the other hand the c-pattern of σ according to Carlitz (1) is a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers such that

$$\lambda_1 + \lambda_2 + ... + \lambda_m = n \qquad (1)$$

$$\sigma(1)<\sigma(2)< \dots < \sigma(\lambda_1),\ \sigma(\lambda_1+1)<\sigma(\lambda_1+2)<\dots<\sigma(\lambda_1+\lambda_2),\dots$$

$$\sigma(\lambda_1+\lambda_2+\dots\lambda_{m-1}+1) < \dots < \sigma(n) \tag{2}$$

$$\sigma(\lambda_1)>\sigma(\lambda_{1+1}),\ \sigma(\lambda_1+\lambda_2)>\sigma(\lambda_1+\lambda_2+1),\dots \tag{3}$$

For example the c-pattern of the permutation $\sigma = 67135428$ is $\lambda = (2,3,1,2)$.

In this paper three different type of sets are studied and it is proved that the cardinality of each of them is equal to the number of Fibonacci.

The first is the set of all patterns which do not contain double rise, the second is the set of all c-patterns with coordinated 1 or/and 2 the third is the set of all permutations σ in S_n the elements of which satisfy certain unequality properties. Moreover a construction of the elements of the third set is given.

2. Propositions

Let V_{n-1} be the set of all patterns $w=z_1,z_2\dots z_{n-1}$ which do not contain double rise, i.e. if $z_i=+$ the z_{i+1} $=-$, $i \in [n-2]$. The following proposition characterizes the cardinality of V_{n-1}.

Proposition 1 : The numbers $|V_{n-1}|$, $n=1,2,\dots$ are the numbers of Fibonacci.

Proof

If A (resp. B) is the set of all $z_1z_2\dots z_{n-1}=w\in V_{n-1}$

such that $z_{n-1} = +$ (resp. $z_{n-1} = -$), then $|V_{n-1}| = |A| + |B|$.

Moreover since the pattern of each $w \in V_{n-1}$ does not contain a double rise we deduce that $z_{n-2} = -$ for each $w \in A$. Thus by considering the following bijections :

$\Phi_1 : V_{n-3} \to A$, with $\Phi_1(z_1 z_2 \ldots z_{n-3}) = z_1 z_2 \ldots z_{n-3} - +$

$\Phi_2 : V_{n-2} \to B$, with $\Phi_2(z_1 z_2 \ldots z_{n-2}) = z_1 z_2 \ldots z_{n-2} -$

We deduce that $|V_{n-3}| = A$ and $|V_{n-2}| = B$.
Thus,

$$|V_{n-1}| = |V_{n-2}| + |V_{n-3}|$$

for each $n = 3,4,\ldots,$ and $|V_0| = 0$, $|V_1| = 1$ i.e. the Fibonacci numbers. For the set F_n of all c-patterns $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_i \in \{1,2\}$, $i \in [n]$, we have the following result.

Proposition 2 : The numbers $|F_n|$, $n=0,1,2,\ldots$ are the numbers of Fibonacci.

Proof

Let A(resp. B) be the set of all $(\lambda_1, \ldots, \lambda_m) = \lambda \in F_n$ such that $\lambda_m = 1$ (resp. 2).
Then $|F_n| = |A| + |B|$
Moreover since the set A (resp. B) is equivalent to the set F_{n-1} (resp. F_{n-2}) we deduce that

$$|A| = |F_{n-1}| \text{ and } |B| = |F_{n-2}|$$

Thus,

$$|F_n| = |F_{n-1}| + |F_{n-2}|$$

for each n=2,3,... and $|F_0| = 0$, $|F_1| = 1$ i.e. the Fibonacci numbers. Now let A_n, $n > 2$ be the set of all permutations $\sigma \in S_n$ for which

$$|\sigma(i) - i| \leq 1$$

and $|A_0| = |A_1| = 1$. Then we have the following result.

Proposition 3 : The numbers $|A_n|$, n=0,1,2,... are the Fibonacci numbers.

Proof

From the definition of the set A_n follows that either $\sigma(n)=n$ or $\sigma(n)=n-1$ and $\sigma(n-1)=n$, for every $\sigma \in A_n$.
Then if we define the sets

$$A_n^1 = \{\sigma \in A_n : \sigma(n) = n\}$$

$$A_n^2 = \{\sigma \in A_n : \sigma(n) = n\text{-}1 \text{ and } \sigma(n-1) = n\}$$

we deduce that

$$|A_n| = |A_n^1| + |A_n^2|$$

$$|A_n^1| = |A_{n-1}| \text{ and } |A_n^2| = |A_{n-2}|$$

It follows that

$$|A_n| = |A_{n-1}| + |A_{n-2}|$$

for every n=2,3,... and the numbers $|A_n|$, n = 0,1,2,... are the Fibonacci numbers.

Finally the determination of the elements of the set A_n is given with the aid of the following sets (2), (3).

$$\Delta_1 = \{1,2\},\ \Delta_n = \{n-1, n\},\ \Delta_i = \{i-1, i, i+1\}\ \forall\ i \neq 1,n$$

Γ(j) : The subset of [n] with elements the terminal vertices of arcs with initial vertex j.

E(j) : The subset of [n], the elements of which are vertices of the path with first vertex 0 and last vertex j.

Proposition 4 : The paths of the permutation tree T_n which satisfy the following conditions :

(i) For the first 0 :

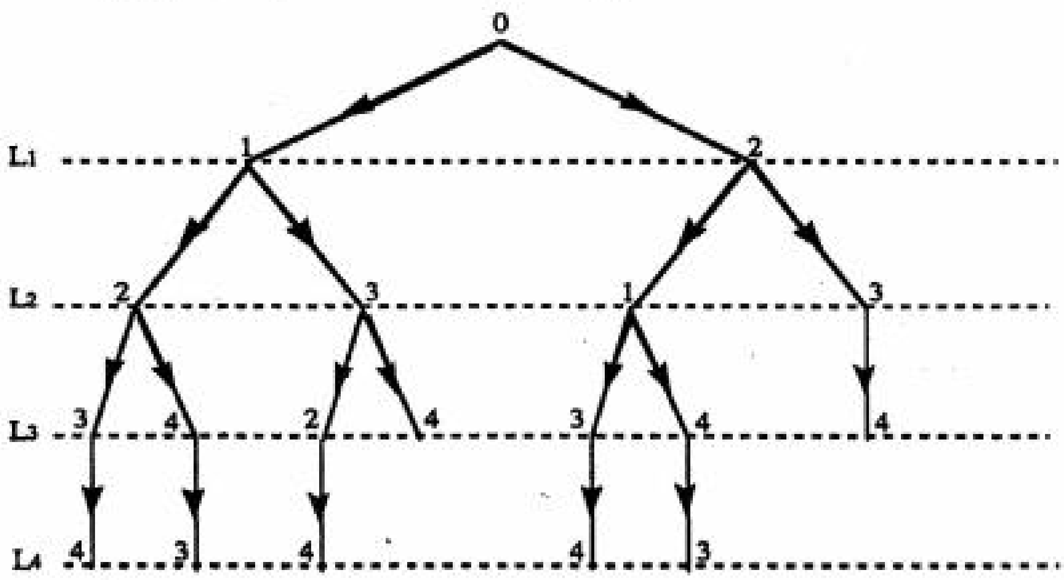
$$\Gamma(0) = \{1,2\}$$

(ii) $\Gamma(j) = ([n] - E(j)) \cap \Delta_{i+1}$ for every level and every j give all the elements of A_n

The first elements of the permutations are produced by the relationship (i). On the other hand the remaining elements of the permutations are produced by the repeated process of relationship (ii).

Example

For n=4 we obtain the following permutation tree.



for every level L_1, L_2, L_3, L_4 and every vertex j, we get the following subsets :

L_1 : $\Gamma(1) = (\{1,2,3,4\} - \{1\}) \cap \{1,2,3\} = \{2,3\}$,...

L_2 : $\Gamma(4) = (\{1,2,3,4\} - \{1,2\}) \cap \{2,3,4\} = \{3,4\}$,...

L_3 : $\Gamma(3) = (\{1,2,3,4\} - \{1,3,4\}) \cap \{3,4\} = \emptyset$,...

: $\Gamma(3) = (\{1,2,3,4\} - \{1,2,3\}) \cap \{3,4\} = \{4\}$,...

Therefore $A_4 = \{1234, 1243, 1324, 2134, 2143\}$

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