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ON THE DISTRIBUTION OF THE PRESENT VALUE OF INTEREST IN CONTINUOUS COMPOUNDING

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1. Introduction

Present value models play a very important role in financial theory and management. The literature has for the main part concentrated on purely deterministic models aimed at the construction of explicit present value formulae under conditions of certainty [6].

Zinn et all [15] and Delbaen et al [7] have considered present value models under conditions of uncertainty. Most of the stochastic treatments of present value models concentrate only on the establishment of mean value, variance and in some cases, semi–variance Young et al [14] and Rosenthal [11]. Stochastic methods which fully exploit the properties and, under certain assumptions, derive the distribution function of a present value model, are the most powerful tools of the theory of random discounting.

Artikis and Voudouri [1] have considered the present value of a single random cash flow under random timing and they hane provided sufficient conditions for the distribution of this present value to belong to the class of α–unimodal distributions. Artikis and Jerwood [4] have studied an extension of Artikis and Voudouri 's present value model to consider the case where timing is a geometric random sum. The new present value model has interesting applications in risk management, replacement policies and capital budgeting.

Another extension of Artikis and Voudouri 's present value model has been studied by Artikis et al [5]. This model is the present value of a Binomial random sum under exponentially distributed timing. The authors have also provided applications of this present value model in investment replacement.

Artikis and Malliaris [3] have used the present value of an arbitarily distributed random sum under exponentially distributed timing to characterize a self–decomposable distribution. Applications of the characterization in investment financing have been provided by the authors.

The present value of a random uniform cash flow of random duration has been studied by Voudouri [13].The author gives sufficient conditions for the distribution of this present value model to be member of the class of ν–unimodal distributions.

The purpose of the present paper is to establish properties and applications of a present value model. Section two introduces a stochastic present value model arising in interest discounting. Section three is devoted to the distribution of the present value model under arbitrarily distributed timing. In section four, the distribution of the present value model under exponentially distributed timing is considered. It is shown that the distribution belongs to the class of ν–unimodal distributions. Moreover, this section provides, certain explicit forms for the characteristic function of the distribution function of the present value model. Section five is devoted to the evaluation of the moments of the present value model under arbitrarily distributed timing. More precisely, it is shown that the moments of the present value are related to the moments of the cash flow and to the characteristic function of the timing. Section six presents an extension of the present value model. The new present value model, which contains a cash flow decomposed into a random sum, has interesting practical applications in financial management.

2. A Stochastic Model Arising in Finance

Let X represent an amount compounded continuously over the time interval [0, T]. The future value U of the X at time T is given by

$$U = X e^{rT}, \; r > 0 \qquad (2.1)$$

where r is the nominal, continuous–compound interest rate [10]. From (2.1), it follows that

$$S = X\,(e^{rT} - 1) \qquad (2.2)$$

is the interest earned during this time interval and received at time T. Moreover

$$V = S\, e^{-rT}$$

or equivalently

$$V = X\,(1 - e^{-rT}) \qquad (2.3)$$

is the present value of S at time 0.

The quantity

$$W = 1 - e^{-rT} \tag{2.4}$$

is called the discount factor of interest in continuous compounding.

When X and T are random variables and r is a constant, the analyst must determine the distribution function of the present value V from the distribution function of X and the distribution function of T. Any analytical determination of the distribution function of the present value V of interest in continuous compounding is intrinsically difficult. However, it will be shown that the distribution of V with T exponentially distributed is related to the class of ν–unimodal distributions [8]. We shall infer that it is the discount factor of interest in continuous compounding with T exponentially distributed, which is primarily responsible for the introduction of the property of ν–unimodality into the distribution function of the present value V in (2.3).

The present paper is devoted to the determination of the distribution function of V, when X and T are continuous positive and independent random variables. Properties of the distribution of V, when T is exponentially distributed with parameter 2r, have been established by Artikis et al [2].

3. Characteristic Function of the Present Value Under Arbitrarily Distributed Time

This section deals with the distribution function of the present value V. It is useful to transform the distribution functions of the random variables using the corresponding characteristic functions [9].

Let T be a continuous positive random variable with probability density function $f_T(t)$ and let

$$W = 1 - e^{-rT}$$

be the discount factor of interest in continuous compounding. The probability density function of W is easily demonstrated to be

$$f_W(w) = \frac{1}{r\,(1-w)}\, f_T\left(-\frac{1}{r}\log\,(1-w)\right),\ 0 \le w \ge 1 \tag{3.1}$$

Let X be a continuous positive random variable with characteristic function $\varphi_X(u)$. If the random variables X and T are independent, then the random variables

$$X, \qquad W = 1 - e^{-rT}$$

are also independent. Hence, the characteristic function of the present value

$$V = X(1 - e^{-rT})$$

is given by

$$\varphi_V(u) = \int_0^1 \varphi_X(uw)\, f_W(w)\, dw$$

$$= \int_0^1 \varphi_X(uw) \frac{1}{r(1-w)} f_T\left(-\frac{1}{r}\log(1-w)\right) dw \qquad (3.2)$$

This result is similar to the result for the characteristic function of the present value of a continuous uniform cash flow, under arbitrarily distributed time, established by Voudouri, [13].

From (3.2), it follows that the distribution of V is very complicated when the random variable T is arbitrarily distributed.

4. Characteristic Function of the Present Value Under Exponentially Distributed Time

If the random variable T is exponentially distributed with probability density function

$$f_T(t) = \lambda e^{-\lambda t}, \; t > 0, \; \lambda > 0$$

then from (3.1) and (3.2), it easily follows that the characteristic function of the present value V is given by

$$\varphi_V(u) = \nu \int_0^1 \varphi_X(uw)\,(1-w)^{\nu-1}\, dw \qquad (4.1)$$

with $\nu = \lambda/r$ and hence the distribution of V belongs to the class of ν–unimodal distributions [8]. This class contains the class of distributions having a unique mode at zero and the class of distributions having convex probability density functions [13].

Moreover, the class of ν–unimodal distributions is the rule than the exception in the present value of interest in continuous compounding since the exponential distribution is the most commonly encountered distribution of time. The discount factor

$$W = 1 - e^{-rT}$$

is responsible for the introduction of the property of v–unimodality in the distribution of the present value of interest in continuous compounding.

Below we consider two particular cases of (4.1). If the characteristic function of the amount X is given by

$$\varphi_X(u) = e^{iu},$$

then from (4.1) with $\nu = 2$, we get that

$$\varphi_V(u) = \frac{2\left(1 + iu - e^{u}\right)}{u^2}$$

which is the characteristic function of the Betta distribution with probability density function

$$f_V(v) = 2(1 - v),\ 0 \leq v \leq 1$$

However, if we have an amount X, which follows the Gamma distribution with characteristic function

$$\varphi_X(u) = \left(\frac{\mu}{\mu - iu}\right)^3, \quad \mu > 0$$

then from (4.1), with $\nu = 2$, we get that

$$\varphi_V(u) = \frac{\mu}{\mu - iu},$$

which is readily identified as the characteristic function of the exponential distribution, with scale parameter μ.

5. Moments of the Present Value

If T is not exponentially distributed, the distribution of the present value

$$V = X\left(1 - e^{-rT}\right)$$

is very complicated. Under the single assumption of independence between the amount X and the time T, then we can write for the moments of the present value V

$$\begin{aligned} E(V^n) &= E(X^n)\, E\left[(1 - e^{-rT})^n\right] \\ &= E(X^n)\, E\left[\sum_{\kappa=0}^{n} (-1)^{\kappa} \binom{n}{\kappa} e^{-\kappa r T}\right] = \\ &= E(X^n) \sum_{\kappa=0}^{n} (-1)^{\kappa} \binom{n}{\kappa} E\left(e^{-\kappa r T}\right) \end{aligned} \tag{5.1}$$

Let

$$\varphi_T(u) = E\left(e^{iuT}\right) \tag{5.2}$$

be the characteristic function of the random variable T. From (5.1) and (5.2), it follows that the moments of V are related to those of X by the result

$$E(V^n) = E(X^n) \sum_{\kappa=0}^{n} (-1)^{\kappa} \binom{n}{\kappa} \varphi_T(i\kappa r) \tag{5.3}$$

and the practical and financial implications have to be interpreted.

The usefulness of the mean value μ_v and the variance σ_v^2 of the present value of interest in continuous compounding to managers is evident. Moreover certain higher moments of the present value V can be useful to managers. In particular, the third moment is important: as pointed out by Stone [12], of all uncertain present worths with equal mean values and variances, the one with maximum skewness is preferred. Positive or right skewness is viewed as desirable, negative or left skewness as undesirable. Formulae for computing skewness of present value of random cash flows under random timing can be derived easily.

6. An Extension of the Present Value Model

When discounting the interest in continuous compounding, it is often unrealistic to assume that there exists a single amount X compounded continously over the time interval [0, T]. More likely scenario arises when the amount X is decomposed into a random sum of amounts. In order to incorporate this feature into the present value of interest in continuous compounding, we define the amounts $\{Y_n: n = 1, 2...\}$ as a sequence of continuous, positive, independent and identically distributed random variables with common distribution function $F_Y(y)$ and characteristic function $\varphi_Y(u)$. The number N of amounts is a discrete random variable with range space {0, 1...} and probability generating function $P_N(z)$. This random variable is taken to be independent of the amounts $\{Y_n: n= 1, 2...\}$. Under these assumptions the amount X, which is continuously compounded over the time interval [0, T] is the random sum

$$X = Y_1 + Y_2 + ... + Y_N.$$

Consequently the characteristic function of X is related to its components by the functional relationship

$$\varphi_X(u) = P_N(\varphi_Y(u)) \tag{6.2}$$

The distribution of V, as defined by (2.3) and subject to (6.2), in the general situation is extremely complicated. Moreover, the establishment of an explicit result, similar to that of (5.3), for the evaluation of the moments of the present value model

$$V = [Y_1 + Y_2 + ... + Y_N]\,[1 - e^{-rT}] \tag{6.3}$$

is not possible.

Below we present an application of the present value model (6.3) in risk control. One way to control a particular pure risk is to avoid the property, person, or activity with which the exposure to risk is associated by refusing to assume it even momentarily or abandoning an exposure assumed earlier. Let us suppose a company is offered a portfolio of N new investments at time 0, where N is a discrete random variable. In particular, if the portfolio consists of N new investments of certain type in a given region the we can suppose that N is a Poisson distributed random variable. The cost of each investment is a random variable because each investment could be made according to alternate specifications. We define the costs $\{Y_n: n = 1, 2...\}$ as a sequence of continuous, positive, independent and identically distributed random variables. Moreover the costs $\{Y_n: n = 1, 2...\}$ are taken to be independent of N. The random sum in (6.1) gives the cost of the N investments at time 0. Considering continuous compounding of this cost over the random time interval [0, T] we get that the model in (6.3) gives the present value of interest earned during [0, T]. This present value is important when the company wants of interest earned during [0, T]. This present value is important when the company wants to avoid the exposures to risk associated with the portfolio of investments by refusing to assume this portfolio of investments. In this case, continuous compounding of the cost of the investments over the random time interval [0, T] is almost always an available alternative way of action for the company. Within the context of investment decision making, the random variable T can be interpreted as the time elapsing between two successive investment proposals.

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