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FUNCTIONAL BOUNDEDNESS OF SOME M-COMPLETE m-CONVEX ALGEBRAS

M. Oudadess

E.N.S , B.P 5118 , Takaddoum, Rabat (Morocco)

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Abstract: We first improve a structure theorem due to M. Akkar and then show that any advertibly complete, M–complete and commutative l.m.c.a every element of which is bounded is functionally bounded.

1.Introduction. In the context of Michael's problem ([7], p.53), M. Akkar obtained a structure theorem ([1], théorème fondamental) for locally m–convex algebras (l.m.c.a). Regarding the nature of the problem that is boundedness of multiplicative functionals (characters) topological completeness can be weakend.

We first extend Akkar's result to Mackey–complete l.m.c.a's. These appear to be bornologically inductive limits of Fréchet l.m.c.a's (Theorem 2.1). This allows to show that they are functionally bounded provided they are advertibly complete with bounded elements; whence as corollaries our results of [8], which are actually valid in a more general setting.

Let (E,τ) be a complex algebra endowed with a locally convex topology given by a family $(p_\lambda)_\lambda$ of seminorms. It is said to be a locally m–convex algebra (l.m.c.a, [7]) if its topology can be given by a family of submultiplicative seminorms. The radius of boundedness β, on (E,τ), is difined by $\beta(x) = \ inf\{r > 0 : (r^{-1})^n,\ n = 1, 2, ...,$ is bounded$\}$ with the convention that $inf\,\phi = +\infty$

For all definitions concerning bornological notions, the reader is referred to [5]. Let us however recall that bornological completeness of the Von Neumann bound structure of a Hausdorff locally convex space is equivalent to its Mackey completeness. In this paper , we will use the latter denomination (M–complete in short). Topological (resp. bornological) notions will be referred to as τ–notions (resp. M–notions).

2. Structure theorem.

The proof of the following result goes along the lines of ([1], théorème fondamental). We reproduce the essential of Akkar's proof using new arguments for the completeness of the topologies τ_i.

Theorem 2.1. Any M–complete l.m.c.a. (E,τ) is bornologically an inductive limit (with continuous injections) of Fréchet l.m.c.a's (E_i,τ_i).

Proof. Let $(p_\lambda)_\lambda$ be a family of submultiplicative seminorms defining the topology and $(B_i)_{i\in I}$ a basis of the bound structure. Fix a B_i. For every λ, there is an $M_\lambda > 0$ such that $p_\lambda(x) \ \leq\ M_\lambda$,

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for every x in B_i. Put, for every integer n,

$$p_\lambda(B_i) = sup\{p_\lambda(x) :\ x\ \in\ B_i\},$$

and consider $\Lambda_n^i = \{\lambda \in \Lambda : p_\lambda(B_i) < n\}$. It is clear $\Lambda = \bigcup \Lambda_n^i$. Now put $q_n^i(x) = sup\{p_\lambda(x) : x\ \in \Lambda_n^i\}$. It is a generalized seminorm (not everywhere finite) which is submultiplicative and lower semi continuous (l.s.c). Consider, for every i, the set

$$E_i = \{x \in E :\ q_n^i(x) < +\infty,\ n = 1, 2, ...\}.$$

It is a subalgebra of E. Endowed with the topology given by the family $(q_n^i)_n$, it becomes a metrizable l.m.c.a. It is straighforward to show that (E, τ) is bornologically an inductive limit of (E_i, τ_i)'s.

It remains to show that every (E_i, τ_i) is complete. Let $(x_p)_p$ be a Cauchy sequence in (E_i, τ_i). It is also M-Cauchy for this space is metrizable. But then it is M-Cauchy in (E, τ) which is supposed to be M-complete. Let x be the M-limit of (x_p) in (E, τ). It is also a τ-limit. It is now enough to show that x is in E_i and that the sequence tends to x for τ_i; which is straightforward, using the definition of E_i and the expression of q_n^i in terms of seminorms p_λ.

As a consequence of the previous results, we have the following

Proposition 2.2. Michael's problem is equivalent to the same problem in an M-complete l.m.c.a .

A second consequence is an improvement of a result due to M. Akkar.

Proposition 2.3. Every positive functional on any unitary and M-complete l.m.c.a with a continuous involution is bounded.

3. Functional boundedness.

A net $(x_\mu)_\mu$, in a unitary topological algebra E is said to be advertibly convergent if there is an x in E such that the nets $(x.x_\mu)_\mu$ and $(x_\mu.x)_\mu$ are both convergent to the unit element e . The algebra is said to be advertibly complete if any Cauchy advertibly convergent sequence is convergent ([6]).

If the algebra is a l.m.c.a, denote by $\mathcal{M}^\#$ (resp. $\mathcal{M}$) the set of nonzero algebraic (resp. continous) charactors of E.

Theorem 3.1. Let (E, τ) be a unitary, commutative, advertibly complete and M-complete l.m.c.a every element of which is bounded. Then $\mathcal{M}^\#$ is equibounded.

Proof. Since every element is bounded its radius $\beta(x)$ is finite. On the other hand , if ρ stands for the spectral radius, then $\beta(x) = \rho(x)$ ([3], Theorem 3.12) for E being M-complete is necessarily pseudo-complete. Notice also that β is subadditive. Moreover, since (E, τ) is advetibly complete , we have $Spx = \{g(x) : g \in \mathcal{M}\}$ (cf. [6]). Hence $\rho(x) = sup\{|g(x)| : g \in \mathcal{M}\}$ is lower semi-continous (l.s.c). Now let R_i be the restriction of ρ to E_i (cf. the structure theorem). This is a seminorm on E_i. It is l.s.c for $R_i=\rho o f_i$, where the injection f_i from E_i to E is continuous and ρ is l.s.c. But E_i is Fréchet hence R_i is continuous. It is then bounded. Whence the boundedness of ρ and hence the equiboundedness of $\mathcal{M}^{\#}$.

Remark 3.2. In [2], we find the following result (Théorème IV.1): If E is a commutative Fréchet l.m.c.a which is not a Q–algebra, then there exists an advertibly complete subalgebra with a discontinuous character. Hence advertible completeness is not sufficient in theorem 3.1

4.Others properties.

Exactly as in [8], we obtain the following results.

Proposition 4.1. Let (E,τ) be a unitary, commutative, advertibly complete and M–complete l.m.c.a.

(1) Every element of (E,τ) is bounded if, and only if, (E,τ) is Mackey Q–algebra

(2) The following are equivalent.

(a) Spx is compact, for every x in E.

(b) $(E,\tau)$is a Mackey Q–algebra.

(c) $\mathcal{M}^{\#}$ is compact in $E^{\#}$, the algebraic dual, for the weak topology.

Concerning the Von Neumann bound structure $I\!B\tau$ of (E,τ) in relation with pseudo–Banach structures, we have the

Proposition 4.2. Let (E,τ)be a unitary, commutative, advertibly complete and M–complete l.m.c.a. Then

(1) The inverse map $x \longmapsto x^{-1}$ is sequentially M–continuous.

(2) The following are equivalent.

(i) $(E,\ I\!B\tau)$ is pseudo–Banach.

(ii)The inverse map $x \longmapsto x^{-1}$ is Mackey continuous.

Remark 4.3. In [2], we find the following example. Consider the algebra $C_c(I\!R^+)$ of continuous complex functions, on $I\!R^+$, which are constant for x larger than a non negative real number. Endowed with the topology of uniform convergence on compact subsets of $I\!R^+$, it becomes a non complete metrizable l.m.c.a. It is advertibly complete. Every element is bounded, but its Von Neumann bound structure is not pseudo–Banach. This shows that we have to put an additional condition (M–completeness here), for the assertion (2) of proposition 4.2 to make sense.

5. On a theorem of Klee.

An ordered locally convex space is said to be locally solid if it is a lattice and its topology can be given by a family $(p_\lambda)_\lambda$ of seminorms satisfying $p_\lambda(x) \le p_\lambda(y)$ whenever $|x| \le |y|$ (cf. [9]). By the same proof as in section 2, we improve a structure theorem on locally solid l.c.s spaces, due to M. Akkar.

Theorem 5.1. Any locally solid and M–complete l.c.s is bornologically an inductive limit (with continuous injections) of locally solid Fréchet l.c.s's.

V.L. Klee has shown that any positive functional on a Fréchet locally solid l.c.s is bounded (cf. [9]). Using the previous theorem, we obtain the following result.

Proposition 5.2. Any positive functional on a locally solid and M-complete l.c.s is bounded.

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