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On the dual of an ideal and ideal transforms

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Abstract

We consider the dual of an ideal and the ideal transforms of Nagata and Kaplansky in the context of monoids. In particular, we consider Prüfer monoids with respect to a finitary ideal system and generalize the known results for Prüfer domains. These generalizations embrace abstract monoids (without an ideal system), Prüfer v-multiplication domains and Prüfer $*$-multiplication domains for any star-operation of finite type.

1. Introduction

Let D be an integral domain with quotient field K and I a non-zero fractional ideal of D. We consider the fractional ideal

$$I^{-1} = (D : I) = \{x \in K \mid xI \subset D\}$$

and the overrings

$$(I : I) = \{x \in K \mid xI \subset I\}, \quad \mathcal{N}_D(I) = \bigcup_{n \in \mathbb{N}} (D : I^n) \quad \text{and} \quad \Omega_D(I) = \bigcap_{0 \neq a \in I} \bigcup_{n \in \mathbb{N}} a^{-n} D.$$

Usually I^{-1} is called the dual of I, $(I : I)$ is called the ring of multipliers of I, $\mathcal{N}_D(I)$ is called the Nagata transform and $\Omega_D(I)$ is called the Kaplansky transform of D with I. We obviously have

$$(I : I) \subset I^{-1} \subset \mathcal{N}_D(I) \subset \Omega_D(I).$$

The questions whether I^{-1} is a ring and whether it coincides with either $(I : I)$ or $\mathcal{N}_D(I)$ have received considerable attention in recent years (see [FHPR], [F], Chapter III in [FHP], or [HKLM] and the literature cited there).

If D is a Prüfer domain, then each overring of D has a representation as an intersection of localizations of D with respect to those prime ideals which survive in the overring under consideration. For the overrings $(I:I)$ and $\mathcal{N}_D(I)$, the responsible primes were identified in [FHPR], Theorem 4.7 and in [FHP], Theorems 3.2.5 and 3.2.6. If I^{-1} is a ring, the same was done in [FHP], Theorem 3.1.2. An analogue to the latter result in the case of PVMDs (Prüfer v-multiplication domains) was given in [HKLM], Theorem 4.5.

The question whether or not I^{-1} is a ring is a purely multiplicative one. Indeed, since I^{-1} is a D-submodule of K, it is a ring if and only if it is multiplicatively closed. If in addition 2 is not a zero divisor on K/I^{-1}, then the simple identity $2xy = (x+y)^2 - x^2 - y^2$ shows that I^{-1} is a ring if and only if it is closed under squares (that is, $z \in I^{-1}$ implies $z^2 \in I^{-1}$).

In this paper we take the point of view of [HK1] that multiplicative ideal theory should be derived as far as possible without making reference to the additive structure. We show that many of the above-mentioned questions and results concerning $(I:I)$, I^{-1}, $\mathcal{N}_D(I)$ and $\Omega_D(I)$ can be derived in the context of cancellative monoids, equipped with suitable ideal systems. When doing this, we shall obtain slightly stronger results with even simpler proofs than in the case of integral domains. In particular, the results for Prüfer domains and PVMDs are special cases of our theorems concerning r-Prüfer monoids. We do not explicitely state these specializations which we leave to the reader.

This paper is organized as follows. In section 2, we fix our notations and recall some fundamentals from the ideal theory of monoids. In section 3 we investigate I^{-1} and $(I:I)$, first without further assumptions on the monoid D, and then under the additional hypothesis that D is seminormal or (integrally) closed with respect to an ideal system. In the seminormal case we also investigate the connections with $\mathcal{N}_D(I)$. In section 4, we assume that D is an r-Prüfer monoid (for some ideal system r), we present the representation theorems for I^{-1} and $(I:I)$ as intersections of localizations and some of their consequences. Finally, in section 5, we investigate $\mathcal{N}_D(I)$ and $\Omega_D(I)$ in more detail.

2. Preliminaries on Monoids

Our main reference for monoids and ideal systems is [HK1]. We recall the most important concepts and fix the notations. We denote by $\mathbb{N}$ the set of positive integers, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set X, we denote by $\mathbb{P}(X)$ its power set and by $\mathbb{P}_f(X)$ the set of all finite subsets of X.

By a monoid D we always mean a commutative multiplicative semigroup possessing a unit element $1 \in D$ (such that $1a = a$ for all $a \in D$), a zero element $0 \in D$ (such that $0a = 0$ for all $a \in D$), and satisfying the cancellation law (if $a, b, c \in D$ and $ab = ac$, then either $a = 0$ or $b = c$). We set $D^\bullet = D \setminus \{0\}$ and denote by $D^\times$ the group of invertible elements of D. By a groupoid we mean a monoid D such that

$D^\bullet$ is a group. Every monoid D possesses a quotient groupoid K, that is, a groupoid $K \supset D$ such that $K = \{a^{-1}b \mid a \in D^\bullet,\ b \in D\}$. By an overmonoid of D we always mean a monoid between D and K. For $X, Y \subset K$ and $n \in \mathbb{N}$, we set

$$XY = \{xy \mid x \in X\,,\ y \in Y\}\,, \quad X^n = \{x_1 \cdot \ldots \cdot x_n \mid x_1, \ldots, x_n \in X\}$$

(note that this differs from the usual definition for ideals in a ring),

$$(X : Y) = (X : Y)_K = \{z \in K \mid zY \subset X\}\,, \quad X^{-1} = (D : X) \quad \text{and} \quad X_v = (X^{-1})^{-1}\,.$$

Recall that a finitary ideal system r on D is a map $\mathbb{P}(D) \to \mathbb{P}(D)$, $X \mapsto X_r$ such that the following conditions are fulfilled for all subsets X, Y of D and all $c \in D$:

1) $X \cup \{0\} \subset X_r$;
2) $X \subset Y_r$ implies $X_r \subset Y_r$;
3) $(cX)_r = cX_r$;
4) $X_r = \bigcup_{E \in \mathbb{P}_f(X)} E_r$.

For a finitary ideal system r on D, we denote by $\mathcal{I}_r(D) = \{J \subset D \mid J_r = J\}$ the semigroup of all r-ideals, equipped with the r-multiplication defined by $I \cdot_r J = (IJ)_r$. For any subset $X \subset K$, we define

$$X_r = \bigcup_{E \in \mathbb{P}_f(X)} E_r \tag{1}$$

(note that, for every finite subset $E \subset K$, there exists some $c \in D^\bullet$ such that $cE \subset D$, and then $E_r = c^{-1}(cE)_r$). Then r is a module system on K in the sense of [HK2]. An overmonoid $T \supset D$ is called an r-monoid if $T_r = T$. For valuation monoids, this notion coincides with that of [HK1], Definition 18.4 (see Lemma 2.3 below). If $T \supset D$ is an r-monoid, then the extension $r[T]$ of r to T is defined by

$$X_{r[T]} = (TX)_r \quad \text{for all} \quad X \subset K\,.$$

Then $r[T]$ is a finitary ideal system on T, and $\mathcal{I}_{r[T]}(T) = \{J \subset T \mid J_r = J$ and $TJ = J\}$ (see [HK2], Section 2).

We denote by r-spec(D) the set of all prime r-ideals and by r-max(D) the set of all r-maximal r-ideals of D. For $P \in s$-spec(D) and $X \subset K$, we denote by

$$X_P = \{s^{-1}x \mid x \in X\,,\ s \in D \setminus P\} \subset K$$

the localization of X with respect to P. If r is a finitary ideal system on D, then D_P is an r-monoid, and we denote by $r_P = r[D_P]$ the localization of r with respect to P.

The most important finitary ideal systems we are concerned with are the s-system of ordinary semigroup ideals and the t-system, defined by

$$E_s = ED \quad \text{and} \quad E_t = (D : (D : E)) \quad \text{for all} \quad E \in \mathbb{P}_f(K)\,,$$

and then extended by (1) to arbitrary subsets $X \subset K$ (for details see [HK1]). Every overmonoid $T \supset D$ is an s-overmonoid, and $s[T]$ is just the s-system on T.

If D is an integral domain with quotient field K, then (disregarding the additive structure) D is a monoid with quotient groupoid K. For a subset $X \subset D$, let X_d be the D-ideal generated by X. Then d is a finitary ideal system on D, called the ideal system of ordinary ring ideals. An overmonoid $T \supset D$ is a d-monoid if and only if it is an overring, and in this case $d[T]$ is just the ordinary d-system on T.

Let D be a monoid and $I \subset D$ an s-ideal. We denote by $\mathcal{P}(I)$ the set of all prime divisors of I (that is, the minimal prime s-ideals containing I), by

$$Z_D(I) = \{a \in D \mid ax \in I \ \text{ for some } \ x \in D \setminus I\}$$

the prime s-ideal of all zero divisors on I and by

$$\sqrt{I} = \{x \in D \mid x^n \in I \text{ for some } n \in \mathbb{N}\} = \bigcap_{P \in \mathcal{P}(I)} P$$

the radical of I (see [HK1], 6.7).

For a finitary ideal system r on D, we denote by $\mathcal{V}_r(D)$ the set of all r-valuation overmonoids of D, we set

$$\mathcal{V}_r(I) = \{V \in \mathcal{V}_r(D) \mid V \setminus V^\times = \sqrt{IV}\} \quad \text{and} \quad \mathcal{W}_r(I) = \{V \in \mathcal{V}_r(D) \mid IV = V\}.$$

We denote by $\mathbf{Z}_r(I)$ the set of all maximal elements (with respect to inclusion) in the set

$$\{P \in r\text{-spec}(D) \mid P \subset Z_D(I)\}.$$

We say that I has no embedded r-primes, if $\{P \in r\text{-spec}(D) \mid I \subset P \subset Z_D(I)\} \subset \mathcal{P}(I)$. Note that $I \in \mathcal{I}_r(D)$ implies $\mathcal{P}(I) \subset r\text{-spec}(D)$ (see [HK1], Proposition 6.6).

For sake of completeness we recall the definition of the Nagata transform and the Kaplansky transform in a purely multiplicative context.

Definition 2.1 Let D be a monoid and $I \subset D$ an s-ideal. Then we set

$$\mathcal{N}_D(I) = \bigcup_{n \in \mathbb{N}} (I^n)^{-1} \qquad \text{and} \qquad \Omega_D(I) = \bigcap_{0 \neq a \in I} \bigcup_{n \in \mathbb{N}} a^{-n} D.$$

We call $\mathcal{N}_D(I)$ the *Nagata transform* and $\Omega_D(I)$ the *Kaplansky transform* of D with respect to I.

We start by collecting some elementary properties of $(I : I)$, I^{-1}, $\mathcal{N}_D(I)$ and $\Omega_D(I)$.

Proposition 2.2 *Let D be a monoid, $I \subset D$ an s-ideal and r a finitary ideal system on D.*

1. $(I:I) \subset I^{-1} = (I^{-1})_r \subset \mathcal{N}_D(I) \subset \Omega_D(I)$.

2. *$(I:I)$, $\mathcal{N}_D(I)$ and $\Omega_D(I)$ are overmonoids of D, $\mathcal{N}_D(I)$ and $\Omega_D(I)$ are r-monoids, and if I is an r-ideal, then $(I:I)$ is also an r-monoid.*

Proof. It follows immediately from the definitions that $(I:I) \subset I^{-1} \subset \mathcal{N}_D(I) \subset \Omega_D(I)$, and that $(I:I)$, $\mathcal{N}_D(I)$ and $\Omega_D(I)$ are overmonoids of D. The remaining assertions concerning I^{-1} and $(I:I)$ follow by [HK1], Proposition 11.7. $\mathcal{N}_D(I)$ and $\Omega_D(I)$ are r-monoids by [HK2], Propositions 1.2 and 2.3. □

We close this introductory section with four simple lemmas concerning valuation monoids and localizations, for which we give proofs because of a lack of a suitable reference. Recall that a monoid V is a valuation monoid if the set of s-ideals of V is a chain.

Lemma 2.3 *If r is a finitary ideal system on a monoid D and $V \supset D$ is a valuation overmonoid, then V is an r-monoid if and only if $E_r \subset EV$ for all $E \in \mathbb{P}_f(D)$.*

Proof. If $V_r = V$ and $E \in \mathbb{P}_f(D)$, then $EV = aV$ for some $a \in E$ and therefore $E_r \subset (EV)_r = (aV)_r = aV_r = aV = EV$.

Suppose now that $E_r \subset EV$ for all $E \in \mathbb{P}_f(D)$. By definition, it suffices to prove that $F_r \subset V$ for all $F \in \mathbb{P}_f(V)$. If $F \in \mathbb{P}_f(V)$, let $c \in D^\bullet$ be such that $cF \subset D$. Then we obtain $F_r = c^{-1}(cF)_r \subset c^{-1}(cFV) = FV \subset V$. □

Lemma 2.4 *If V is be a valuation monoid and $Q \in s\text{-spec}(D)$, then $QV_Q = Q_Q = Q$.*

Proof. By definition, $QV_Q = Q_Q \supset Q$. Thus, uppose that $x \in Q_Q$, say $x = a^{-1}b$, where $b \in Q$ and $a \in V \setminus Q$. By [HK1], Theorem 16.2, we obtain $Q = aQ$ and therefore $x = a^{-1}b \in a^{-1}aQ = Q$. □

Lemma 2.5 *If D is a monoid, $I \in \mathcal{I}_s(D)$, $P \in s\text{-spec}(D)$ and $I \not\subset P$, then $I^{-1} \subset D_P$.*

Proof. If $a \in I^{-1}$ and $x \in I \setminus P$, then $ax \in D$ and therefore $a = x^{-1}ax \in D_P$. □

Lemma 2.6 *If r is a finitary ideal system on a monoid D, $I \in \mathcal{I}_r(D)$, $P \in s\text{-spec}(D)$, and if Ω denotes the set of all maximal elements (with respect to inclusion) in the set $\{Q \in r\text{-spec}(D) \mid Q \subset P\}$, then*

$$D_P = \bigcap_{Q \in \Omega} D_Q \qquad \text{and} \qquad (I_P : I_P) = \bigcap_{Q \in \Omega} (I_Q : I_Q)\,.$$

Proof. By [HK1], Theorem 11.3, Theorem 7.2 and Ex. 4.6, we have

$$I_P = \bigcap_{M \in r_P\text{-max}(D_P)} (I_P)_M ,$$

r_P-max$(D_P) = \{Q_P \mid Q \in \Omega\}$ and $(I_P)_{Q_P} = I_Q$ for all $Q \in \Omega$. With $I = D$, the first assertion follows. For the second one, we calculate

$$(I_P : I_P) = \Big(\bigcap_{Q \in \Omega} (I_P)_{Q_P} : I_P \Big) = \bigcap_{Q \in \Omega} ((I_P)_{Q_P} : I_P)) .$$

For $Q \in \Omega$, we have $\big((I_P)_{Q_P} : I_P\big) = \big((I_P)_{Q_P} : (I_P)_{Q_P}\big) = (I_Q : I_Q)$, which completes the proof. □

3. The Dual of an Ideal

Throughout this section, let D be a monoid and K a quotient groupoid of D.

Definition 3.1 A subset $X \subset K$ is called *power-closed*, if $x \in X$ implies $x^n \in X$ for all $n \in \mathbb{N}$.

Proposition 3.2 *Let $I \subset D$ be an s-ideal such that I^{-1} is power-closed.*

1. *$I^{-1} = (\sqrt{I} : I)$, and if $I \neq \{0\}$, then $I^{-1} \subset \widehat{D}$ (the complete integral closure of D).*
2. *For all $P \in s$-spec(D), $I \subset P$ implies $I^{-1} = (P : I)$.*
3. *If $V \in \mathcal{V}_s(I)$, then $I^{-1} \subset V$.*

Proof. 1. It is obviously sufficient to prove that $I^{-1} \subset (\sqrt{I} : I)$. If $x \in I^{-1}$, then $x^2 \in I^{-1}$, and $(xI)^2 = (x^2I)I \subset I$ implies $xI \subseteq \sqrt{I}$. Whence $x \in (\sqrt{I} : I)$.
If $x \in I^{-1}$ and $0 \neq c \in I$, then $cx^n \in D$ for all $n \in \mathbb{N}$ and therefore $x \in \widehat{D}$.

2. By 1., we obtain $I^{-1} \subset (\sqrt{I} : I) \subset (P : I)$. The other inclusion is obvious.

3. Assume to the contrary that there exsists some $V \in \mathcal{V}_s(I)$ such that $I^{-1} \not\subset V$. If $x \in I^{-1} \setminus V$, then $x^{-1} \in V \setminus V^\times = \sqrt{IV}$, and therefore $(x^{-1})^n \in IV$ for some $n \in \mathbb{N}$. Since $x^n \in I^{-1}$ we obtain, using 1., that $1 = x^n(x^{-1})^n \in I^{-1}IV \subset \sqrt{I}V \subset \sqrt{IV}$, a contradiction. □

Proposition 3.3 *Let $I \subset D$ be an s-ideal such that I^{-1} is power-closed.*

1. *Let $J \subset D$ be an s-ideal and $S \subset D^\bullet$ a multiplicatively closed subset such that $I \subset J \subset \sqrt{S^{-1}I}$ and $\sqrt{S^{-1}J} \cap D = J$. Then $J^{-1} = (J : J)$.*

2. $\sqrt{I}^{-1} = (\sqrt{I} : \sqrt{I})$.

3. *If $P \in \mathcal{P}(I)$, then $P^{-1} = (P : P)$.*

Proof. 1. It suffices to prove that $J^{-1} \subset (J : J)$. If $x \in J^{-1}$ and $a \in J$, then $J \subset \sqrt{S^{-1}I}$ implies $sa^n \in I$ for some $s \in S$ and $n \in \mathbb{N}$. Since $J^{-1} \subset I^{-1}$ and I^{-1} is power-closed, we obtain $x^{2n} \in I^{-1}$ and $s(ax)^{2n} = a^n(sa^nx^{2n}) \in J$. Whence $ax \in \sqrt{S^{-1}J} \cap D = J$.

2. By 1. with $J = \sqrt{I}$ and $S = \{1\}$.

3. By 1. with $J = P$ and $S = D \setminus P$ (observe that $\sqrt{I_P} = P_P \supset P$). □

Corollary 3.4 *If I is an s-ideal of D satisfying $\sqrt{I} = I$, then the following statements are equivalent:*

(a) $I^{-1} = (I : I)$.

(b) *$I^{-1} \supset D$ is an overmonoid of D.*

(c) *I^{-1} is power-closed.*

(d) *If $x \in I^{-1}$ then $x^2 \in I^{-1}$.*

Proof. Obviously, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (a). It suffices to prove that $I^{-1} \subset (I : I)$. If $x \in I^{-1}$, then $(xI)^2 = (x^2I)I \subset I$ implies $xI \subset \sqrt{I} = I$ and therefore $x \in (I : I)$. □

Proposition 3.5 *If r is a finitary ideal system on D, $I \in \mathcal{I}_r(D)$ and $I \neq D$, then the following assertions are equivalent:*

(a) *I^{-1} is an overmonoid of D.*

(b) *I is not r-invertible, and $(P : I)$ is an overmonoid of D for each $P \in r\text{-max}(D)$ such that $I \subset P$.*

(c) *$I^{-1} = (\sqrt{I} : I)$, and $(P : I)$ is an overmonoid of D for each $P \in \mathcal{P}(I)$.*

(d) $I^{-1} = (I_v : I_v)$.

(e) $I^{-1} = (II^{-1} : II^{-1})$.

The assertions (a), (b) *and* (c) *remain equivalent if we replace "overmonoid" by "power-closed".*

Proof. (a) $\Leftrightarrow$ (d). By [HK1], Proposition 13.6, observing that $I^{-1} = I_v^{-1}$.

(a) $\Leftrightarrow$ (e). By [HK1], Ex. 13.2.

(a) $\Rightarrow$ (b), (c). By Proposition 3.2.1, we have $I^{-1} = (\sqrt{I} : I)$, and consequently $I \cdot_r I^{-1} = (II^{-1})_r \subset (\sqrt{I})_r = \sqrt{I} \subsetneq D$. Whence I is not r-indvertible. If $P \in r\text{-spec}(D)$ and $P \supset I$, then $(P : I) = I^{-1}$ by Proposition 3.2.2.

(b) $\Rightarrow$ (a). Since I is not r-invertible, there exists some $P \in r\text{-max}(D)$ such that $II^{-1} \subset I \cdot_r I^{-1} \subset P$. Whence $I^{-1} \subset (P:I) \subset I^{-1}$. Therefore $I^{-1} = (P:I)$ is an overmonoid.

(c) $\Rightarrow$ (a). If $P \in \mathcal{P}(I)$, then $\sqrt{I} \subset P$, and therefore $I^{-1} = (\sqrt{I}:I) = (P:I) \subset I^{-1}$. Whence $I^{-1} = (P:I)$ is an overmonoid of D. □

Next we investigate the dual of an ideal under the additional assumption that D is seminormal. Recall that D is called seminormal if the following condition is satisfied: If $z \in K$ and $z^n \in D$ for all sufficiently large $n \in \mathbb{N}$, then $z \in D$.

Proposition 3.6 *Let D be seminormal and $I \subset D$ an s-ideal of D.*

1. *We have*

$$(\sqrt{I}:\sqrt{I}) = \{x \in K \mid x^n \in I^{-1} \textit{ for all } n \in \mathbb{N}\}$$
$$= \{x \in K \mid x^n \in I^{-1} \textit{ for all sufficiently large } n \in \mathbb{N}\}.$$

2. *I^{-1} is power-closed if and only if $I^{-1} = (\sqrt{I}:\sqrt{I})$.*

3. *If I^{-1} is power-closed, then $I^{-1} = (\sqrt{I})^{-1}$.*

Proof. 1. We must prove: **1)** If $x \in (\sqrt{I}:\sqrt{I})$, then $x^n \in I^{-1}$ for all $n \in \mathbb{N}$. **2)** If $x \in K$ and $x^n \in I^{-1}$ for all sufficiently large $n \in \mathbb{N}$, then $x \in (\sqrt{I}:\sqrt{I})$.

1) If $x \in (\sqrt{I}:\sqrt{I})$ and $n \in \mathbb{N}$, then $x^n \in (\sqrt{I}:\sqrt{I})$ and therefore $x^n I \subset x^n\sqrt{I} \subset \sqrt{I} \subset D$. Whence $x \in I^{-1}$.

2) Suppose that $x \in K$, $x^n \in I^{-1}$ for all sufficiently large $n \in \mathbb{N}$ and $a \in \sqrt{I}$. Then we have $a^n \in I$ and hence $(ax)^n \in D$ for all sufficiently large $n \in \mathbb{N}$. Since D is seminormal, it follows that $ax \in D$. For sufficiently large $n \in \mathbb{N}$, we obtain $(ax)^{n+1} = (a^n x^{n+1})a \in aD \subset \sqrt{I}$ and therefore $ax \in \sqrt{I}$.

2. Obvious by 1.

3. Suppose that I^{-1} is power-closed. It is sufficient to prove that $I^{-1} \subset (\sqrt{I})^{-1}$. If $x \in I^{-1}$ and $a \in \sqrt{I}$, then $x^n \in I^{-1}$, $a^n \in I$ and hence $(ax)^n \in D$ for all sufficiently large $n \in \mathbb{N}$. Since D is seminormal, we infer $ax \in D$, and therefore $x \in (\sqrt{I})^{-1}$. □

With the aid of Proposition 3.6 we are able to investigate the connection between I^{-1} and $\mathcal{N}_D(I)$ in the seminormal case.

Proposition 3.7 *If D is seminormal and I is an s-ideal of D, then the following assertions are equivalent:*

(a) $I^{-1} = \mathcal{N}_D(I)$.

(b) $(I^n)^{-1}$ *is an overmonoid of* D *for all* $n \in \mathbb{N}$.

(c) $(I^n)^{-1}$ *is an overmonoid of* D *for some* $n \geq 2$.

In (b) and (c) the phrase "is an overmonoid of D" can be replaced by "is power-closed".

Proof. (a) $\Rightarrow$ (b). For $n \in \mathbb{N}$, we have $\mathcal{N}_D(I) = I^{-1} \subset (I^n)^{-1} \subset \mathcal{N}_D(I)$. Hence $(I^n)^{-1} = \mathcal{N}_D(I)$ is an overmonoid of D.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a). Since $I^n \subset I^2 \subset I \subset \sqrt{I}$, Proposition 3.6.2 implies

$$(I^n)^{-1} = \left(\sqrt{I^n} : \sqrt{I^n}\right) = (\sqrt{I} : \sqrt{I}) \subset (\sqrt{I})^{-1} \subset I^{-1} \subset (I^2)^{-1} \subset (I^n)^{-1},$$

and therefore $(I^2)^{-1} = I^{-1}$. Now we obtain $(I^m)^{-1} = I^{-1}$ by induction on m. Indeed, if $m \geq 2$ and $(I^m)^{-1} = (I^{m-1})^{-1}$, then

$$(I^{m+1})^{-1} = \left((I^m)^{-1} : I\right) = \left((I^{m-1})^{-1} : I\right) = (I^m)^{-1}.$$

By Proposition 3.6.2, $(I^n)^{-1}$ is an overmonoid if and only if it is power-closed. □

We close this section with an investigation of $(I : I)$ and I^{-1} under the additional assumption that D is r-closed with respect to a finitary ideal system r on D. We recall the basic facts on generalized integral closures (see [HK1], Ch. 14).

Let r be a finitary ideal system on D. We say that D is r-closed if $(J : J) = D$ for all r-finitely generated r-ideals of D. Note that D is s-closed if and only if it is root-closed, and if D is an integral domain, then it is d-closed if and only if it is integrally closed. If D is r-closed, then D is root-closed and hence seminormal.

Suppose that D is r-closed. Then the finitary ideal system r_a (called the completion of r) is defined by

$$X_{r_a} = \bigcup_{\substack{B \in \mathbb{P}_f(D) \\ B \cap D^\bullet \neq \emptyset}} ((XB)_r : B) \qquad \text{for all subsets} \quad X \subset D.$$

By [HK1], Theorem 21.7, we have

$$X_{r_a} = \bigcap_{V \in \mathcal{V}_r(D)} XV \qquad \text{for all } D\text{-fractional subsets} \quad X \subset K,$$

and in particular

$$D = \bigcap_{V \in \mathcal{V}_r(D)} V.$$

If $X \subset D$ and $V \in \mathcal{V}_r(D)$, then $X_{r_a}V = XV$ (see [HK1], Theorem 21.3).

Lemma 3.8 *Let r be a finitary ideal system on D such that D is r-closed, and let $I \subset D$ be an s-ideal.*

1. *We have*

$$(I_{r_a} : I_{r_a}) = \bigcap_{V \in \mathcal{V}_r(D)} (IV : IV).$$

2. *If $\mathcal{V} \subset \mathcal{V}_r(D)$ is any subset, then*

$$D = \bigcap_{V \in \mathcal{V}} V \quad \textit{implies} \quad \bigcap_{V \in \mathcal{V}_r(I)} V \cap \bigcap_{W \in \mathcal{W}_r(I) \cap \mathcal{V}} W \subset I^{-1} \subset \bigcap_{W \in \mathcal{W}_r(I) \cap \mathcal{V}} W.$$

Proof. 1. Observe that

$$(I_{r_a} : I_{r_a}) = \Big(\bigcap_{V \in \mathcal{V}_r(D)} IV : I_{r_a} \Big) = \bigcap_{V \in \mathcal{V}_r(D)} (IV : I_{r_a}).$$

For each $V \in \mathcal{V}_r(D)$, we have $(IV : I_{r_a}) = (IV : I_{r_a}V) = (IV : IV)$. Thus the assertion follows.

2. Assume that D is the intersection of the valuation monoids $V \in \mathcal{V}$, and let first x be an element of the intersection on the left hand side. It is sufficient to prove that $a \in I$ implies $ax \in U$ for all $U \in \mathcal{V}$. Thus suppose that $a \in I$ and $U \in \mathcal{V}$. If $U \in \mathcal{W}_r(I)$, then $x \in U$ implies $ax \in U$. If $U \in \mathcal{V}_r(D) \setminus \mathcal{W}_r(I)$, then $Q = \sqrt{IU} \in s\text{-spec}(U)$, and since $\sqrt{IU_Q} = Q_Q = U_Q \setminus U_Q^\times$, we obtain, using Lemma 2.4, $U_Q \in \mathcal{V}_r(I)$ and $ax \in IU_Q \subset Q_Q = Q \subset U$.

If $x \in I^{-1}$ and $W \in \mathcal{W}_r(I) \cap \mathcal{V}$, then $x \in xW = xIW \subset W$. □

The following Theorem 3.9 generalizes [HKLM], Theorem 4.4.

Theorem 3.9 *Let r be a finitary ideal system on D such that D is r-closed, and let $\mathcal{V} \subset \mathcal{V}_r(D)$ be a subset such that*

$$D = \bigcap_{V \in \mathcal{V}} V.$$

If I is an s-ideal of D, then the following assertions are equivalent:

(a) *I^{-1} is an overmonoid of D.*

(b) *I^{-1} is power-closed.*

(c) *$I^{-1} \subset (II^{-1}V : II^{-1}V)$ for each $V \in \mathcal{V}_r(D)$.*

(d) *$I^{-1} \subset (I_vV : I_vV)$ for each $V \in \mathcal{V}_r(D)$.*

(e) *There exists some $J \in \mathcal{I}_s(D)$ such that $J \supset I$ and $I^{-1} \subset (JV : JV)$ for each $V \in \mathcal{V}_r(D)$.*

(f) *$I^{-1} \subset V$ for each $V \in \mathcal{V}_r(I)$.*

(g) *We have the representation*

$$I^{-1} = \bigcap_{V \in \mathcal{V}_r(I)} V \cap \bigcap_{W \in \mathcal{W}_r(I) \cap \mathcal{V}} W .$$

Proof. (a) $\Rightarrow$ (c), (d). By Proposition 3.5 we obtain $I^{-1} = (II^{-1} : II^{-1}) \subset (II^{-1}V : II^{-1}V)$ and $I^{-1} = (I_v : I_v) \subset (I_vV : I_vV)$ for all $V \in \mathcal{V}_r(D)$.

(c), (d) $\Rightarrow$ (e). Obvious.

(e) $\Rightarrow$ (a). By Lemma 3.8, we obtain

$$I^{-1} \subset \bigcap_{V \in \mathcal{V}_r(D)} (JV : JV) = (J_{r_a} : J_{r_a}) \subset J_{r_a}^{-1} \subset J^{-1} \subset I^{-1} ,$$

and therefore $I^{-1} = (J_{r_a} : J_{r_a})$ is an overmonoid of D.

(a) $\Rightarrow$ (b) is obvious, (b) $\Rightarrow$ (f) follows by Proposition 3.2.3, (f) $\Rightarrow$ (g) follows by Lemma 3.8 (2), and (g) $\Rightarrow$ (a) is again obvious. □

4. The Dual of an Ideal in r-Prüfer monoids

Throughout this section, let D be a monoid and r a finitary ideal system on D.

Recall that D is called an r-Prüfer monoid if D_P is a valuation monoid for each $P \in r\text{-spec}(D)$. For several other characterizations see [HK1], Ch. 17. If D is an r-Prüfer monoid, then every overmonoid of D is an intersection of localizations of D (see [HK1], Theorem 27.2). We want to specify the responsible prime ideals in the representation of $(I : I)$ and of I^{-1} (provided that it is an overmonoid). We start with a technical lemma concerning the local behavior of $\mathcal{Z}_D(I)$.

Lemma 4.1 *Suppose that $I \in \mathcal{I}_s(D)$, $P \in s\text{-spec}(D)$ and $I \subset P$. Then we have*

$$\mathcal{Z}_D(I_P \cap D) = \mathcal{Z}_{D_P}(I_P) \cap D \qquad \textit{and} \qquad \mathcal{Z}_{D_P}(I_P) = \mathcal{Z}_D(I_P \cap D)_P .$$

Proof. It is sufficient to prove the first equality (see [HK1], Theorem 4.4). If $a \in \mathcal{Z}_D(I_P \cap D)$, then there exists some $z \in D \setminus I_P \subset D_P \setminus I_P$ such that $az \in I_P$. Whence $a \in \mathcal{Z}_{D_P}(I_P) \cap D$.

Suppose now that $a \in \mathcal{Z}_{D_P}(I_P) \cap D$. Then there exists some $z \in D_P \setminus I_P$ such that $az \in I_P$. If $s \in D \setminus P$ is such that $sz \in D$, then $sz \notin I_P$, and $asz \in I_P \cap D$ implies $a \in \mathcal{Z}_D(I_P \cap D)$. □

The following Theorem 4.2 generalizes [FHPR], Theorems 4.7 and 4.11, and [HKLM], Theorem 4.7.

Theorem 4.2 *Let D be r-Prüfer.*

1. *If I is an s-ideal of D, $P \in r\text{-spec}(D)$ and $I \subset P$, then $\mathcal{Z}_D(I_P \cap D) \in r\text{-spec}(D)$, and*
$$(I_P : I_P) = D_{\mathcal{Z}_D(I_P \cap D)}\,.$$

2. *For each s-ideal $I \subset D$, we have the representation*
$$(I : I) \;=\; \left(I_{\mathcal{Z}_D(I)} : I_{\mathcal{Z}_D(I)}\right) \;\cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M\,.$$

3. *For each r-ideal $I \in \mathcal{I}_r(D)$, we have the representations*
$$\left(I_{\mathcal{Z}_D(I)} : I_{\mathcal{Z}_D(I)}\right) = \bigcap_{P \in \mathbf{Z}_r(I)} (I_P : I_P)$$
and
$$(I : I) \;=\; \bigcap_{\substack{M \in r\text{-max}(D) \\ I \subset M}} (I_M : I_M) \;\cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M\,.$$

4. *For each s-ideal $I \subset D$ without embedded r-primes, we have the representation*
$$(I : I) \;=\; (\sqrt{I} : \sqrt{I}) \;=\; D_{\mathcal{Z}_D(I)} \;\cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M\,.$$

Proof. 1. Since $\mathcal{Z}_{D_P}(I_P) \in s\text{-spec}(D_P) = r_P\text{-spec}(D_P)$, we infer (using Lemma 4.1) that $\mathcal{Z}_D(I_P \cap D) = \mathcal{Z}_{D_P}(I_P) \cap D \in r\text{-spec}(D)$. By [HK1], Ex. 15.8 and Ex. 4.6, we obtain
$$(I_P : I_P) = (D_P)_{\mathcal{Z}_{D_P}(I_P)} = D_{\mathcal{Z}_{D_P}(I_P) \cap D} = D_{\mathcal{Z}_D(I_P \cap D)}\,.$$

2. For each $P \in s\text{-spec}(D)$ we have $(I : I) \subset (I_P : I_P)$, and if $I \not\subset P$, then $(I_P : I_P) = (D_P : D_P) = D_P$. Hence it remains to prove that
$$\left(I_{\mathcal{Z}_D(I)} : I_{\mathcal{Z}_D(I)}\right) \;\cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M \;\subset\; (I : I)\,.$$

Let z be an element of the intersection on the left hand side, and $a \in I$. We must prove that $za \in I$. Since $za \in I_{\mathcal{Z}_D(I)}$, there exists some $t \in D \setminus \mathcal{Z}_D(I)$ such that $tza \in I$. We prove first that $za \in D$, and for that, it is sufficient to prove that $za \in D_M$ for all $M \in r\text{-max}(D)$.

If $M \in r\text{-max}(D)$ and $I \not\subset M$, then $z \in D_M$ implies $az \in D_M$. Thus suppose that $M \in r\text{-max}(D)$, $I \subset M$ and $za \notin D_M$. Then we have $(za)^{-1} \in D_M$ and

$t = tza(za)^{-1} \in I_M$. Therefore there exists some $u \in D \setminus M$ such that $tu \in I$, and $t \notin \mathcal{Z}_D(I)$ implies $u \in I \subset M$, a contradiction.

Now $tza \in I$, $za \in D$ and $t \in D \setminus \mathcal{Z}_D(I)$ implies $za \in I$, as asserted.

3. We apply Lemma 2.6. To obtain the first representation, we set $P = \mathcal{Z}_D(I)$. To obtain the second one, we set $P = D \setminus D^\times$ and observe that $(I_M : I_M) = (D_M : D_M) = D_M$ for all $M \in r\text{-max}(D)$ such that $I \not\subset M$.

4. By Lemma 2.6, we have

$$\bigcap_{P \in \mathbf{Z}_r(I)} D_P = D_{\mathcal{Z}_D(I)} \subset \left(I_{\mathcal{Z}_D(I)} : I_{\mathcal{Z}_D(I)}\right).$$

Hence it suffices by 2. to prove that $(I : I) \subset (\sqrt{I} : \sqrt{I}) \subset D_P$ for all $P \in \mathbf{Z}_r(I)$.

If $x \in (I : I)$, then $x^n \in (I : I) \subset I^{-1}$ for all $n \in \mathbb{N}$, and Proposition 3.6.1 implies $x \in (\sqrt{I} : \sqrt{I})$.

If $P \in \mathbf{Z}_r(I)$ and $I \not\subset P$, then $\sqrt{I} \not\subset P$, and therefore $(\sqrt{I} : \sqrt{I}) \subset (\sqrt{I}_P : \sqrt{I}_P) = (D_P : D_P) = D_P$. If $P \in \mathbf{Z}_r(I)$ and $I \subset P$, then $P \in \mathcal{P}(I)$ by assumption and therefore $P_P = \sqrt{I_P} = \sqrt{I}_P$, which implies $(\sqrt{I} : \sqrt{I}) \subset (\sqrt{I}_P : \sqrt{I}_P) = (P_P : P_P) = D_P$ by [HK1], Ex. 15.8. □

Corollary 4.3 *If D is r-Prüfer, $I \in \mathcal{I}_r(D)$ and $n \in \mathbb{N}$, then $(I : I) = (I^n : I^n)$.*

Proof. It is easily seen that $(I : I) \subset (I^n : I^n)$. To prove the reverse inclusion, assume first that D is a valuation monoid. If $u \in (I^n : I^n)$, then $u^m \in (I^n : I^n)$ for all $m \in \mathbb{N}$, and in particular $(uI)^n \subset I^n$. If $a \in I$, then $(ua)^n \in I^n \subset D$ implies $ua \in D$, and therefore $ua \in I$ by [HK1], Ex. 15.6. Hence $u \in (I : I)$ follows.

For the general case, we apply Theorem 4.2.3 and observe that $\left((I^n)_r\right)_M = I^n_M$ for all $M \in r\text{-max}(D)$. Thus we obtain

$$(I^n : I^n) \subset \left((I^n)_r : (I^n)_r\right) = \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} (I^n_M : I^n_M) \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M$$

$$= \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} (I_M : I_M) \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M = (I : I).$$

□

For sake of completeness, we recall the (well known) criterion for I^{-1} to be an overmonoid in the case of valuation monoids.

Proposition 4.4 *If D is a valuation monoid and $I \subsetneq D$ a (proper) s-ideal of D, then I^{-1} is an overmonoid of D if and only if I is a non-principal prime s-ideal. In this case, we have $I^{-1} = (I : I) = D_I$.*

Proof. [HK1], Ex. 16.8. □

Lemma 4.5 *Let D be r-Prüfer and $I \subset D$ an s-ideal.*

1. $\mathcal{P}(I_r) \subset \mathcal{P}(I)$.
2. *We have*

$$\bigcap_{P \in \mathcal{P}(I)} D_P \ \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M \ \subset \bigcap_{P \in \mathcal{P}(I_r)} D_P \ \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M \ \subset \ I^{-1}.$$

Proof. 1. Suppose that $P \in \mathcal{P}(I_r)$. Then $P \in r$-spec(D), and there exists some $Q \in \mathcal{P}(I)$ such that $Q \subset P$ (see [HK1], Proposition 6.6). D_P is a valuation monoid, and Q_P is a prime s-ideal of D_P. Since D_P possesses only one finitary ideal system, Q_P is an r_P-ideal. Whence $Q = Q_P \cap D \in r$-spec(D) (see [HK1], Theorems 15.3, 4.4 and 7.2). Thus we obtain $I_r \subset Q \subset P$, and consequently $Q = P \in \mathcal{P}(I_r)$.

2. We apply 1. and Lemma 3.8 with $\mathcal{V} = r$-max(D) (see [HK1], Theorem 11.3). Since

$$\mathcal{V}_r(I_r) = \{D_P \mid P \in \mathcal{P}(I_r)\}$$

and

$$\mathcal{W}_r(I_r) \cap \mathcal{V} = \{D_M \mid M \in r\text{-max}(D)\,,\ I \not\subset M\} = \{D_M \mid M \in r\text{-max}(D)\,,\ I_r \not\subset M\}\,,$$

we obtain

$$\bigcap_{P \in \mathcal{P}(I)} D_P \ \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M \ \subset \bigcap_{P \in \mathcal{P}(I_r)} D_P \ \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M \ \subset I^{-1}.$$

□

The following Theorem 4.6 is a common generalization of [FHP], Theorem 3.1.2 and [HKLM], Theorem 4.5.

Theorem 4.6 *If D is r-Prüfer and $I \subset D$ is an s-ideal, then the following assertions are equivalent:*

(a) *I^{-1} is an overmonoid of D.*

(b) *I^{-1} is power-closed.*

(c) *$I^{-1} \subset D_P$ for all $P \in \mathcal{P}(I)$.*

(d) *We have the representation*

$$I^{-1} = \bigcap_{P \in \mathcal{P}(I_r)} D_P \ \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M\,.$$

(e) *We have the representation*

$$I^{-1} = \bigcap_{P \in \mathcal{P}(I)} D_P \ \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M .$$

Proof. Obviously, (a) $\Rightarrow$ (b), (d) $\Rightarrow$ (a) and (e) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c). Let I^{-1} be power-closed and $P \in \mathcal{P}(I)$.

CASE 1: $P_r \neq D$. If $Q \in \mathcal{P}(P_r)$, then $Q \in r\text{-spec}(D)$ and therefore D_Q is a valuation monoid. Now $P \subset Q$ implies $D_Q \subset D_P$, hence D_P is a valuation monoid (see [HK1], Corollary 15.1), and $D_P \setminus D_P^\times = P_P = \sqrt{I_P} = \sqrt{ID_P}$. Therefore we obtain $D_P \in \mathcal{V}_s(I)$, and $I^{-1} \subset D_P$ by Proposition 3.2

CASE 2: $P_r = D$. Since r is finitary, there exists a finite subset $E \subset P$ such that $E_r = D$. Since $E \subset P_P = \sqrt{I_P} \subset D_P$, there exists some $n \in \mathbb{N}$ and $s \in D \setminus P$ such that $sx^n \in I$ for all $x \in E$. Suppose now that $u \in I^{-1}$, and set $E^{[n]} = \{x^n \mid x \in E\}$. Then we have $usE^{[n]} \subset D$ and therefore $us \in usD = us(E^{[n]})_r \subset D$ (see [HK1], Ex. 12.3). Hence $u \in D_P$.

(c) $\Rightarrow$ (d), (e). By Lemma 4.5.2 and Lemma 2.5. □

We close this section with two special criteria for I^{-1} to be an overmonoid of D. Proposition 4.8.1 is a partial generalization of [FHP], Theorem 3.1.7, and Proposition 4.8.2 generalizes [FHP], Corollary 3.1.8. This latter mentioned assertion will be strengthened in Theorem 5.7. Proposition 4.9 is a common generalization of [FHPR], Theorem 4.11, and [HKLM], Theorem 4.7 and Corollary 4.8. First we need an elementary lemma concerning r-invertible ideals in r-Prüfer monoids.

Lemma 4.7 *If D is r-Prüfer and $P \in r\text{-spec}(D) \setminus r\text{-max}(D)$, then P is not r-invertible.*

Proof. Assume to the contrary that there exists an r-invertible $P \in r\text{-spec}(D)$ and some $M \in r\text{-max}(D)$ such that $P \subsetneq M$. Then $P = E_r$ for some finite subset $E \subset P$. If $z \in M \setminus P$, then $P \subsetneq (E \cup \{z\})_r$, and since D is r-Prüfer, $(E \cup \{z\})_r$ is r-invertible. But this contradicts [HK1], Theorem 13.2. □

Proposition 4.8 *Let D be r-Prüfer.*

1. *Suppose that $I \in \mathcal{I}_r(D)$, $\sqrt{I} = I$ and $\mathcal{P}(I) \cap r\text{-max}(D) = \emptyset$. Then I^{-1} is an overmonoid of D.*

2. *If $P \in r\text{-spec}(D)$ is not r-invertible, then $P^{-1} = (P : P)$, and if $P \in r\text{-max}(D)$, then $P^{-1} = D$.*

Proof. 1. By Theorem 4.6, it is sufficient to prove that $I^{-1} \subset D_P$ for all $P \in \mathcal{P}(I)$. Thus let $P \in \mathcal{P}(I)$ and $M \in r\text{-max}(D)$ be such that $P \subsetneq M$. Then $P_M \subsetneq M_M \subset D_M$, and $P_M = \sqrt{I_M} = (\sqrt{I})_M = I_M$. By Lemma 4.7, P_M is not r-invertible, and by Proposition 4.4 we obtain

$$I^{-1} \subset (D_M : I_M) = (D_M : P_M) = (D_M)_{P_M} = D_P .$$

2. Let $P \in r\text{-spec}(P)$ be not r-invertible.

If $P \notin r\text{-max}(D)$, then P^{-1} is an overmonoid of D by 1., Theorem 4.6 implies

$$P^{-1} = D_P \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ P \not\subset M}} D_M ,$$

and by Theorem 4.2.2 we get $P^{-1} = (P : P)$.

If $P \in r\text{-max}(D)$, then $P \subset (PP^{-1})_r \subsetneq D$ implies $P = (PP^{-1})_r \supset PP^{-1}$, hence $P^{-1} \subset (P : P)$, and therefore $P^{-1} = (P : P)$ is an overmonoid of D. Now Theorem 4.6 implies

$$P^{-1} = D_P \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ M \neq P}} D_M = D .$$

□

Proposition 4.9 *Let D be r-Prüfer.*

1. *If I is an s-ideal of D without embedded r-primes such that I^{-1} is an overmonoid of D and*

$$\mathcal{Z}_D(I) = \bigcup_{P \in \mathcal{P}(I)} P ,$$

then $I^{-1} = (I : I)$.

2. *If q is a finitary ideal system on D such that $\mathcal{I}_r(D) \subset \mathcal{I}_q(D)$, and if $M \in q\text{-max}(D)$ is not q-invertible, then $M^{-1} = D$.*

Proof. 1. By Theorem 4.6 we have

$$I^{-1} = \bigcap_{P \in \mathcal{P}(I)} D_P \cap \bigcap_{\substack{M \in r\text{-max}(D) \\ I \not\subset M}} D_M ,$$

and Lemma 2.6 implies

$$\bigcap_{P \in \mathcal{P}(I)} D_P = D_{\mathcal{Z}_D(I)} .$$

Now the assertion follows by Theorem 4.2.4.

2. Since $M \subset (MM^{-1})_q \subsetneq D$, we obtain $M = (MM^{-1})_q \supset MM^{-1}$ and therefore $M^{-1} \subset (M : M)$. Hence $M^{-1} = (M : M)$ is an overmonoid. If M is an r-ideal, then M is not r-invertible, and the assertion follows by Proposition 4.8. If M is not an r-ideal, then $M_r = D$ and $M^{-1} = M_r^{-1} = D$. □

5. The Nagata and the Kaplansky Transform

Throughout this section, let D be a monoid and r a finitary ideal system on D.

The results of this section are well known for Prüfer domains (see [FHP], Chapter III). We start with a representation of the Kaplansky transform as an intersectin of localizations, which is valid in every monoid.

Proposition 5.1 *Let $I \subset D$ be an r-ideal.*

1. *If I is r-finitely generated, then $\mathcal{N}_D(I) = \Omega_D(I)$.*
2. *We have the representation*

$$\Omega_D(I) = \bigcap_{\substack{P \in r\text{-spec}(D) \\ I \not\subset P}} D_P .$$

Proof. 1. Suppose that $I = E_r$ for some finite subset $E \subset I$. We must prove that $\Omega_D(I) \subset \mathcal{N}_D(I)$. If $x \in \Omega_D(I)$, then there exists some $n \in \mathbb{N}$ such that $a^n x \in D$ for all $a \in E$. If $E^{[n]} = \{a^n \mid a \in E\}$, then $E^{[n]}x \subset D$, and

$$I^{|E|n}x \subset (E^{|E|n})_r x \subset (E^{[n]}D)_r x \subset D .$$

Hence $x \in I^{-|E|n} \subset \mathcal{N}_D(I)$.

2. Suppose that $x \in \Omega_D(I)$ and $P \in r\text{-spec}(D)$ such that $I \not\subset P$. If $a \in I \setminus P$, then there exists some $n \in \mathbb{N}$ such that $a^n x \in D$, and we obtain $x \in a^{-n}D \subset D_P$.

Let now $x \neq 0$ be an element of the intersection on the right hand side, and $J = x^{-1}D \cap D$. Then $J \in \mathcal{I}_r(D)$, and for all $P \in r\text{-spec}(D)$, $I \not\subset P$ implies $J \not\subset P$. Therefore we obtain

$$\sqrt{J} = \bigcap_{\substack{P \in r\text{-spec}(D) \\ P \supset J}} P \supset \bigcap_{\substack{P \in r\text{-spec}(D) \\ P \supset I}} P \supset I .$$

If $a \in I$, then there exists some $n \in \mathbb{N}$ such that $a^n \in J$ and therefore $a^n x \in D$. Hence we obtain $x \in \Omega_D(I)$. □

Proposition 5.2 *Let T be an r-monoid satisfying $D \subset T \subset \Omega_D(I)$.*

1. *The map*

$$\Psi : \begin{cases} \{P \in r\text{-spec}(D) \mid I \not\subset P\} & \to \{Q \in r[T]\text{-spec}(T) \mid I \not\subset Q\} \\ \qquad P & \mapsto \qquad P_P \cap T \end{cases}$$

is bijective. If $Q \in r[T]\text{-spec}(T)$ and $I \not\subset Q$, then $\Psi^{-1}(Q) = Q \cap D$ and $T_Q = D_{Q \cap D}$. If $P \in r\text{-spec}(D)$ and $I \not\subset P$, then $D_P = T_{P_P \cap T}$.

2. *If* $(TI)_r = T$, *then* $T = \Omega_D(I)$.

Proof. 1. If $P \in r$-spec(D) and $I \not\subset P$, then $\Omega_D(I) \subset D_P$ by Proposition 5.1.2, and therefore $P_P \cap T$ is a prime s-ideal of T. We clearly have $(P_P \cap T) \cap D = P$ and hence $I \not\subset P_P \cap T$. Since $(P_P)_r = P_P$ and $T_r = T$, it follows that $P_P \cap T \in r[T]$-spec(T). Thus we have proved that Ψ is a map as indicated, and that it is injective.

To finish the proof, suppose that $Q \in r[T]$-spec(T) and $I \not\subset Q$. We shall prove that $Q \cap D \in r$-spec(D), $I \not\subset Q \cap D$, $D_{Q\cap D} = T_Q$ and $Q = (Q \cap D)_{Q\cap D} \cap T$.

If $Q \in r[T]$-spec(T) and $I \not\subset Q$, then clearly $P = Q \cap D \in s$-spec(D), $I \not\subset P$, and $Q_r = Q$ implies $P_r = P$. Consequently, $P \in r$-spec(D). Since $D \subset T$ and $D \setminus P \subset T \setminus Q$, we obtain $D_P \subset T_Q$. To prove the reverse inclusion, suppose that $x \in T_Q$, say $x = s^{-1}z$, where $s \in T \setminus Q$ and $z \in T$. Let $a \in I \setminus P$ be arbitrary. Since $s, z \in \Omega_D(I)$, there exists some $n \in \mathbb{N}$ such that sa^n, $za^n \in D$, and $s \notin Q$ implies $sa^n \notin P$. Therefore we get $x = (sa^n)^{-1}(za^n) \in D_P$. Now we set $Q' = P_P \cap T$. As we have just proved, $Q' \in r[T]$-spec(T), $I \not\subset Q'$ and $Q' \cap D = P$. Hence, we obtain $T_{Q'} = D_P = T_Q$, $Q'_{Q'} = Q_Q$, and consequently $Q' = Q'_{Q'} \cap T = Q_Q \cap T = Q$.

2. If $(TI)_r = T$, then $\{Q \in r[T]\text{-spec}(T) \mid I \not\subset Q\} = r[T]$-spec$(T)$, and 1. implies $r[T]$-spec$(T) = \{P_P \cap T \mid P \in r\text{-spec}(D),\ I \not\subset P\}$. Therefore we obtain, using [HK1], Theorem 11.3 and Proposition 5.1.2, that

$$T = \bigcap_{\substack{P \in r\text{-spec}(D)\\ I \not\subset P}} T_{P_P \cap T} = \bigcap_{\substack{P \in r\text{-spec}(D)\\ I \not\subset P}} D_P = \Omega_D(I).$$

□

Before we study the behavior of the Nagata and Kaplansky transform in valuation and Prüfer monoids, we collocate their properties under localizations.

Lemma 5.3 *Let* $I \subset D$ *be an* r*-ideal of* D *and* $P \in s$-spec(D).

1. *We have* $\mathcal{N}_D(I)_P \subset \mathcal{N}_{D_P}(I_P)$, *with equality holding if* I *is* r*-finitely generated.*
2. *If* $I \not\subset P$, *then* $\mathcal{N}_D(I) \subset D_P$, *and if* I *is* r*-finitely generated, then* $\mathcal{N}_D(I)_P = D_P$.
3. $\Omega_D(I) \subset \Omega_{D_P}(I_P)$.

Proof. 1. By [HK1], Proposition 11.7, we have

$$\mathcal{N}_D(I)_P = \bigcup_{n\in\mathbb{N}} (D : I^n)_P \subset \bigcup_{n\in\mathbb{N}} (D_P : I_P^n) = \mathcal{N}_{D_P}(I_P),$$

with equality if I is r-finitely generated.

2. If $I \not\subset P$, then $\mathcal{N}_{D_P}(I_P) = \mathcal{N}_{D_P}(D_P) = D_P$, and the assertion follows by 1.

3. Suppose that $x \in \Omega_D(I)$ and $c = s^{-1}a \in I_P$, where $c \in I$ and $s \in D \setminus P$. Then there exists some $n \in \mathbb{N}$ such that $a^n x \in D$, and consequently $c^n x = s^{-n}a^n x \in D_P$. $\square$

Theorem 5.4 *Let D be a valuation monoid, $I \subsetneq D$ a (proper) s-ideal of D and*

$$Q = \bigcap_{n \in \mathbb{N}} I^n .$$

1. *Q is a prime s-ideal, and $\mathcal{N}_D(I) = D_Q$.*
2. *We have $\mathcal{N}_D(I) = \Omega_D(I)$ if and only if either $I \neq I^2$, or I is the union of all prime s-ideals properly contained in I.*

Proof. By [HK1], Proposition 16.1, we obtain $Q \in s\text{-spec}(D)$, and every prime s-ideal properly contained in I is contained in Q.

1. If $I = I^2$, then $I = I^n = Q$ for all $n \in \mathbb{N}$. Hence $Q^{-1} = \mathcal{N}_D(I)$ is an overmonoid of D, and the assertion follows by Proposition 4.4.

If $I \neq I^2$, then $I \not\subset Q$, and Lemma 5.3.2 implies $\mathcal{N}_D(I) \subset D_Q$. For the reverse inclusion, it is sufficient to prove that $s^{-1} \in \mathcal{N}_D(I)$ for all $s \in D \setminus Q$. If $s \in D \setminus Q$, then $s \notin I^n$ for some $n \in \mathbb{N}$, hence $I^n \subset sD$ and $s^{-1} \in (I^n)^{-1} \subset \mathcal{N}_D(I)$.

2. If $I \neq I^2$, then $\{P \in r\text{-spec}(D) \mid I \not\subset P\} = \{P \in r\text{-spec}(D) \mid P \subset Q\}$, and therefore Proposition 5.1.2 implies $\Omega_D(I) = D_Q = \mathcal{N}_D(I)$.

Suppose next that I is the union of all prime s-ideals properly contained in I. Then we have $Q = I$, and we must prove that $\Omega_D(I) \subset D_Q$. If $u \in \Omega_D(I)$, then $u \in D_P$ for all $P \in s\text{-spec}(D)$ properly contained in I by Propositon 5.1.2. If $u \in D$, we are done. If $u \notin D$, then $u^{-1} \in D$, hence $u^{-1} \in D_P^\times$ and therefore $u^{-1} \notin P$ for all $P \in s\text{-spec}(D)$ properly contained in I. Thus we obtain $u^{-1} \notin I = Q$ and therefore $u = (u^{-1})^{-1} \in D_Q$.

It remains to consider the case where $I = I^2$ and $\mathcal{N}_D(I) = \Omega_D(I)$. Let P^* be the union of all prime s-ideals properly contained in I. Then we have $\mathcal{N}_D(I) = D_I$ by 1. and $\Omega_D(I) = D_{P^*}$ by Proposition 5.1.2. Thus $P^* = I$ follows. $\square$

Theorem 5.5 *Let D be r-Prüfer, $I \subsetneq D$ an r-ideal, $P \in \mathcal{P}(I)$ and*

$$I(P) = \bigcap_{n \in \mathbb{N}} I_P^n \cap D .$$

1. *We have* $I(P) \in r\text{-spec}(D)$, $D_{I(P)} = \mathcal{N}_{D_P}(I_P)$, *and*

$$\mathcal{N}_D(I) \subset \bigcap_{P \in \mathcal{P}(I)} D_{I(P)} \cap \bigcap_{\substack{M \in r\text{-spec}(D) \\ I \not\subset M}} D_M .$$

2. *If the set $\mathcal{P}(I)$ is finite, then*

$$\mathcal{N}_D(I) = \bigcap_{P\in\mathcal{P}(I)} D_{I(P)} \cap \bigcap_{\substack{M\in r\text{-spec}(D)\\ I\not\subset M}} D_M = \bigcap_{P\in\mathcal{P}(I)} D_{I(P)} \cap \bigcap_{\substack{M\in r\text{-max}(D)\\ I\not\subset M}} D_M .$$

Proof. 1. If $P \in \mathcal{P}(I)$, then Theorem 5.4.1 implies

$$I^*(P) = \bigcap_{n\in\mathbb{N}} I_P^n \in s\text{-spec}(D_P) = r_P\text{-spec}(D_P) \quad \text{and} \quad \mathcal{N}_{D_P}(I_P) = (D_P)_{I^*(P)} .$$

Hence we conclude $I(P) \in r$-spec(D), and $I(P) = I^*(P) \cap D$ implies $I^*(P) = I(P)_P$. Thus we obtain $\mathcal{N}_{D_P}(I_P) = D_{I(P)}$, and Lemma 5.3.1 implies $\mathcal{N}_D(I) \subset \mathcal{N}_{D_P}(I_P) = D_{I(P)}$. If $M \in r$-max(D) and $I \not\subset M$, then Lemma 5.3.2 implies $\mathcal{N}_D(I) \subset D_M$.

2. Let $\mathcal{P}(I)$ be finite. We must prove that

$$\bigcap_{P\in\mathcal{P}(I)} D_{I(P)} \cap \bigcap_{\substack{M\in r\text{-max}(D)\\ I\not\subset M}} D_M \subset \mathcal{N}_D(I) .$$

Let u be an element of the given intersection. For each $P \in \mathcal{P}(I)$, there exists an element $s_P \in D \setminus I(P)$ such that $s_P u \in D$. Since $s_P \notin I(P)$, there exists some $n_P \in \mathbb{N}$ such that $s_P \notin I_P^{n_P}$. Now we set $n = \max\{n_P \mid P \in \mathcal{P}(I)\}$, and we obtain $s_P \notin I_P^n$ and therefore $I_P^n \subset s_P D_P$ for all $P \in \mathcal{P}(I)$. This implies $uI_P^n \subset us_P D_P \subset D_P$ for all $P \in \mathcal{P}(I)$, and consequently

$$uI^n \subset \bigcap_{P\in\mathcal{P}(I)} D_P \cap \bigcap_{\substack{M\in r\text{-max}(D)\\ I\not\subset M}} D_M .$$

By Lemma 4.5.2, the intersection on the right hand side is contained in I^{-1}. Hence we obtain $u \in (I^{n+1})^{-1} \subset \mathcal{N}_D(I)$. □

Corollary 5.6 *Let D be r-Prüfer, $P \in r$-spec(D) and*

$$P_0 = \bigcap_{n\in\mathbb{N}} P_P^n \cap D .$$

1. *We have $P_0 \in r$-spec(D) and, for every $Q \in r$-spec(D), $Q \subsetneq P$ implies $Q \subset P_0$.*
2. $\mathcal{N}_D(P) = D_{P_0} \cap \Omega_D(P)$.

Proof. 1. By [HK1], Proposition 16.1, we have

$$P^* = \bigcap_{n\in\mathbb{N}} P_P^n \in s\text{-spec}(D_P) = r_P\text{-spec}(D_P) ,$$

and P^* is the greatest s-ideal of D_P properly contained in P_P. Thus we obtain $P_0 = P^* \cap D \in r\text{-spec}(D)$. If $Q \in r\text{-spec}(D)$ and $Q \subsetneq P$, then $Q_P \subsetneq P_P$, hence $Q_P \subset P^*$ and therefore $Q = Q_P \cap D \subset P^* \cap D = P_0$.

2. By Theorem 5.5.2 (observe that $P(P) = P_0$), and Proposition 5.1.2. □

Theorem 5.7 *Let D be r-Prüfer and $P \in r\text{-spec}(D)$ not r-invertible.*

1. *$P^{-1} = (P : P)$, and there are no r-monoids properly lying between P^{-1} and $\Omega_D(P)$.*
2. *If $P = P^2$, then $P^{-1} = \mathcal{N}_D(P)$*
3. *If P is the union of all prime r-ideals of D properly contained in P, then $P^{-1} = \mathcal{N}_D(P) = \Omega_D(P)$.*
4. *If $P \neq P^2$, then $\mathcal{N}_D(P) = \Omega_D(P)$.*

Proof. 1. By Proposition 4.8.2, we have $P^{-1} = (P : P)$.

Let T be an r-monoid satisfying $P^{-1} \subset T \subset \Omega_D(P)$. If $(PT)_r = T$, then Proposition 5.2.2 implies $T = \Omega_D(P)$. Thus suppose that $P_{r[T]} = (PT)_r \neq T$. By [HK1], Theorem 27.2, T is $r[T]$-Prüfer, and $T \subset D_P$. By Proposition 5.1.2 and Theorem 4.6, we obtain

$$T \subset D_P \cap \bigcap_{\substack{M \in r\text{-spec}(D) \\ I \not\subset M}} D_M = P^{-1},$$

and therefore $T = P^{-1}$.

2. Obvious by the definition.

3. and 4. We set

$$P_0 = \bigcap_{n \in \mathbb{N}} P_P^n \cap D$$

and apply Corollary 5.6.

Suppose first that P is the union of all prime r-ideals of D properly contained in P. Then $P_0 = P$, and P_P is the union of all prime s-ideals of D_P properly contained in P_P. Hence there is no greatest prime s-ideal of D_P properly contained in P_P, and by [HK1], Proposition 16.1, we obtain $P_P = P_P^2$ and therefore $P = P^2$. Hence $P^{-1} = \mathcal{N}_D(P)$ follows by definition. By Lemma 5.3.3 and Theorem 5.4, we obtain

$$\Omega_D(P) \subset \Omega_{D_P}(P_P) = (D_P)_{P_P} = D_P\,.$$

If $P \neq P^2$, then $P_P \neq P_P^2$ and therefore $P_0 \subsetneq P$, and Propositon 5.1.2 implies $\Omega_D(P) \subset D_{P_0}$. □

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