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Special Elements in H_V-structures

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Abstract

The hyperstructures are algebraic structures endowed with at least one hyperoperation. We focus on H_V-structures which are generalized algebraic hyperstructures where, in the axioms of the classical hyperstructures, the equality is replaced by the non-empty intersection. These axioms are called weak. The hyperstructures are classified according to their properties and they correspond to the classical structures by using the fundamental quotients. In the procedure to describe the fundamental relations, special elements appear. These elements are the elements of the core and the so called *single* ones, which are used to define and to study the H_V-structures. In this paper we present some applications on the above argument.

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1. Introduction

The largest class of algebraic hyperstructures we deal with, is the one which satisfies the corresponding structure-like axioms and they are called ***H_v-structures***. They were introduced in [15] and satisfy the weak axioms where the non-empty intersection replaces the equality. In a set H equipped with a hyperoperation $\cdot : H \times H \to \mathcal{P}(H)$, we abbreviate by ***WASS*** the ***weak associativity***:

$$(xy)z \cap x(yz) \neq \emptyset, \ \forall\, x, y, z \in H$$

and by ***COW*** the ***weak commutativity***:

$$xy \cap yx \neq \emptyset, \ \forall\, x, y \in H.$$

A hyperstructure $(H, \cdot)$ is called ***H_v-semigroup*** if it is WASS.

Let $(H, \cdot)$ and $(H, \otimes)$ be two H_v-semigroups defined on the same set H, then $(\cdot)$ is called ***smaller*** than $(\otimes)$, and $(\otimes)$ ***greater*** than $(\cdot)$, iff there is a $f \in Aut(H, \otimes)$ such that

$$xy \subset f(x \otimes y) \ \text{ for all } \ x, y \ \text{ in } \ H$$

then we write $\cdot \leq \otimes$. Moreover, we say that $(H,\otimes)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,\otimes)$ is called ***H_b-structure***.

Remark 1. An important and simple theorem states that for greater hyperoperations of the ones which are WASS or COW then they are also WASS and COW respectively.

A H_v-semigroup $(H,\cdot)$ is called ***H_v-group*** if the reproduction axiom: $xH = Hx = H$, $\forall\, x \in H$, is valid. A hyperstructure $(R,+,\cdot)$ is called ***H_v-ring*** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$, i.e.

$$x(y+z)\cap(xy+xz)\neq\emptyset,\ (x+y)z\cap(xz+yz)\neq\emptyset,\ \forall\, x,y,z\in R.$$

The ***fundamental relations*** β^*, γ^* and ε^* are defined, for example in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalence relations so that the quotient would be the corresponding structure of group, ring and vector space, respectively, these are called *fundamental* (see [1,15,9,4,10,11,8]). The way to find the fundamental classes is given by analogous theorems to the following one [15]:

Theorem 2. Let $(H,\cdot)$ be a H_v-group and let us denote by $\mathcal{U}$ the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset \boldsymbol{u}$ where $\boldsymbol{u}\in\mathcal{U}$. Then the fundamental relation β^* is the transitive closure of the relation β.

The main point of the proof of this theorem is that the relation β is a relation which guaranties the validity of the following: take two elements x,y such that $\{x,y\}\subset \boldsymbol{u}\in \mathcal{U}$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts' of the above β-classes are 'products', then they are fundamental classes.

Let $(H,\cdot)$ be a H_v-group. The class corresponding to the unit element of the fundamental group H/β^*, is called the *core* of the H_v-group and it is denoted by $\boldsymbol{\omega_H}$. Otherwise, core is the kernel of the canonical map $\varphi: H\to H/\beta^*$. An element of a H_v-group is called ***single*** if its fundamental class is a singleton i.e. $\beta^*(x)=\{x\}$ where $\beta^*(x)$ denotes the fundamental class of $x\in H$. We denote by $\boldsymbol{S_H}$ the set of single elements. Analogous definitions are given for the rest H_v-structures.

The relation γ^* in the H_v-ring $(R,+,\cdot)$, is the smallest equivalence such that R/γ^* is a 'ring'. Remark that this 'ring' has not necessarily a unit and $(+)$ is not commutative. If $(+)$ is COW then R/γ^* is a ring, where it does not necessarily contain a unit. If $(\cdot)$ has a unit, then R/γ^* is a ring in the usual sense, because the commutativity in R/γ^* with respect to $(+)$ it is obtained from the rest axioms. However in $(R,+)$ there is no guarantee that $(+)$ is commutative, not even that it is COW. Therefore, generalizing classical structures into hyperstructures, one considers weaker corresponding structures. In order to find the corresponding hyperstructure from a given one, the axioms in the classical theory are normally generalised.

The fundamental relations are also useful to give the general definitions of some hyperstructures. A characteristic application is the general hyperfield where there was no general definition up to 1990 [15], but there is now by using the relation γ^*.

Definition 3. The H_v-ring $(R, +, \cdot)$ is called ***H_v-field*** if the quotient of R by γ^* is a field.

This definition, apart from being very general, gives the idea to introduce a new class of hyperstructures as one can see in the following definition [12]:

Definition 4. A H_v-semigroup $(H, \cdot)$ is called ***h/v-group*** if the quotient H/β^* is a group.

The class of h/v-groups is more general than the class of H_v-groups because in h/v-groups the reproductivity axiom is not necessarily valid. However, sometimes the *reproductivity of classes* is valid. That means that if H is partitioned into equivalence classes $\sigma(x)$, then $x\sigma(y) = \sigma(xy) = \sigma(x)y$, $\forall\ x, y \in H$, is valid [12]. This leads the quotient to be reproductive. Similarly the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined.

For more definitions and results on H_v-structures, see in the book [9] as well as in several papers as [4,5,6,8,10,11,13,14].

In 1989 the *uniting elements* method was introduced by Corsini-Vougiouklis [2]. The main point of this method is that one puts together, in the same class, two or more elements. This leads, through hyperstructures, to structures which satisfy some additional properties. In [5] constructions with desired fundamental structures were introduced and the main point is that the elements of a structure are replaced by sets in such a way that the obtained H_v-structure (or h/v-structure) has the same fundamental structure. This paper has an analogous target by using some special elements.

2. The fundamental classes

A basic, but usually difficult, problem is to find the fundamental classes. There is the following necessary and sufficient condition [9].

Theorem 5. Let $(H, \cdot)$ be an H_v-group, then $u\beta^*u'$ iff there exists $A, A' \subset \beta^*(a)$ and $B, B' \subset \beta^*(b)$ for some $a, b \in H$ such that $uA \cap B \neq \emptyset$ and $u'A' \cap B' \neq \emptyset$.

Proof. Let us suppose that there are $a, b \in H$, $A, A' \subset \beta^*(a)$, and $B, B' \subset \beta^*(b)$ such that $uA \cap B \neq \emptyset$ and $u'A' \cap B' \neq \emptyset$. Then we have

$$\varphi(u) \otimes \varphi(A) \cap \{\varphi(B)\} \neq \emptyset \quad \text{and} \quad \varphi(u') \otimes \varphi(A') \cap \{\varphi(B')\} \neq \emptyset.$$

Where $\otimes$ denotes the 'product' of classes, which is an operation; therefore we obtain

$$\beta^*(u) \otimes \beta^*(a) = \beta^*(b) \quad \text{and} \quad \beta^*(u') \otimes \beta^*(a) = \beta^*(b),$$

which implies $\beta^*(u) = \beta^*(b) \otimes (\beta^*(a))^{-1} = \beta^*(u')$. Thus, $u\beta^*u'$.
The converse is immediate since if $u\beta^*u'$, then we can simply take $A = A' = \omega_H$ and $B = B' = \beta^*(u)$.

Corollary. Let $(R, +, \cdot)$ be an H_v-ring, then $u\gamma^*u'$ iff there exists $A, A' \subset \gamma^*(a)$ and $B, B' \subset \gamma^*(b)$ for some $a, b \in H$ such that $u + A \cap B \neq \emptyset$ and $u' + A' \cap B' \neq \emptyset$.

As sets used in the above theorem and corollary, one can take products of elements of H_v-groups or sums of products of elements of H_v-rings.

3. Enlarging hyperstructures

Constructing several new classes of hyperstructures from the old ones or from the classical ones, the main intention is to keep the original hyperstructures or its fundamental structure as a quotient. This is so, because it is useful to transfer as many as possible of the known properties. A direction to this problem is to take known classes of hyperstructures and to enlarge them by adding elements. We present some of the known results on the topic and for more results one can see for example in [5], [6].

Theorem 6. Let $(G, \cdot)$ be a semigroup and $v \notin G$ be an element appearing in a special product ab, where $a, b \in G$, thus the result becomes a hyperproduct $a \otimes b = \{ab, v\}$. Then the hyperoperation $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $\cdot \leq \otimes$ in the restriction on G, and such that $(G', \otimes)$ is a minimal H_V-semigroup which has fundamental structure isomorphic to $(G, \cdot)$, is defined as follows:

$$a \otimes b = \{ab, v\}, \quad x \otimes y = xy \text{ for all } (x, y) \text{ in } G^2 - \{(a, b)\}$$

$$v \otimes v = abab, \quad x \otimes v = xab \text{ and } v \otimes x = abx \text{ for all } x \text{ in } G.$$

Therefore $(G', \otimes)$ is a *very thin* H_V-semigroup.

Theorem 7. Let $(G, \cdot)$ be a group and $v \notin G$ be an element appearing in a special product ab, where $a, b \in G$, thus the result becomes a hyperproduct $a \otimes b = \{ab, v\}$. Then the $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G, and such that $(G', \otimes)$ is a minimal H_V-group which has fundamental structure isomorphic to $(G, \cdot)$, is defined as follows:
first extend $(\cdot)$ in G' by setting

$$x \cdot v = xab, \quad v \cdot x = abx \quad \text{for all} \quad x \quad \text{in} \quad G, \quad \text{and} \quad v{\bullet}v = abab$$

and then enlarge $(\cdot)$ as follows

(a) for all $(x, y) \in G^2$, for which $xy = ab \neq e$, where e be the unit of $(G, \cdot)$, we set $x \otimes y = \{ab, v\}$ and in the rest cases $x \otimes y = xy$.

(b) If $ab = e$, then, for all $x, y \in G$, for which $xy = e$, set $x \otimes y = \{e, v\}$, moreover, $v \otimes v = \{e, v\}$ and the rest cases $x \otimes y = xy$.

Proof. The WASS is clear since $\cdot \leq \otimes$ and $(\cdot)$ is WASS. In order to have $x \otimes G' = G' \otimes x = G'$ and not to have greater classes, the only ones results we can enlarge are those for which $xy = ab$. On the other hand, the sets $x \cdot G'$ and $G' \cdot x$, must be equal to G', consequently, they have to be enlarged by setting $x \otimes y = \{ab, v\}$. Finally, for the element v in order to have $v \otimes G = G \otimes v = G$ we must have either (a) $e \otimes v = \{ab, v\}$ or (b) in the case $ab = e$ then we set only $v \otimes v = \{e, v\}$.

Remark: if $(G, \cdot)$ is commutative then the $(G', \otimes)$ becomes strongly commutative.

With an analogous proof one can obtain the following in hyperings [6]:

Theorem 8. Let $(R, +, \cdot)$ be a ring and $v \notin R$ be an element appeared in a sum $a + b$, where $a, b \in R$, thus, $a \oplus b = \{a + b, v\}$. Then the $(\oplus)$ and $(\otimes)$ extended in $R' = R \cup \{v\}$ such that $+ \leq \oplus$ and $\cdot \leq \otimes$ in the restriction on R, and such that $(R', \oplus, \otimes)$ is minimal H_V-ring with fundamental ring isomorphic to $(R, +, \cdot)$, is defined as follows:

first extend $(+)$ in R' by setting

$$x + v = x + a + b,\ v + x = a + b + x, \text{ for all } x \text{ in } R, \text{ and } v + v = a + b + a + b$$

$$x \cdot v = x(a + b),\ v \cdot x = (a + b)x \text{ for all } x \text{ in } R, \text{ and } v \cdot v = (a + b)(a + b)$$

and then enlarge $(+)$ as follows

(a) for all $(x, y) \in R^2$, for which $x + y = a + b \neq 0$, where 0 be the zero of $(G, +)$, we set $x \oplus y = \{a + b, v\}$ and in the rest cases $x \oplus y = x + y$.

(b) If $a + b = 0$, then, for all $x, y \in G$, for which $x + y = 0$, set $x \oplus y = \{0, v\}$, moreover, $v \oplus v = \{0, v\}$ and the rest cases $x \oplus y = x + y$.

Moreover, $(\otimes)$ coincides with the extended $(\cdot)$.
It is clear that we have $(R', \oplus, \otimes)/\gamma^* \cong (R, +, \cdot)$.

Constructions in the above sense are the following ones:

The Attach Construction. Let $(H, \cdot)$ be a H_V-semigroup and $v \notin H$. Then we extend the hyperoperation $(\cdot)$ in the set $\underline{H} = H \cup \{v\}$ as follows:

$$x \cdot v = v \cdot x = v \quad \textit{for all} \quad x \in H, \quad \text{and} \quad v \cdot v = H.$$

$(\underline{H}, \cdot)$ is a h/v-group where $(\underline{H}, \cdot)/\beta^* \cong \boldsymbol{Z}_2$ and v is a single element.

We call the hyperstructure $(\underline{H}, \cdot)$ *the attach h/v-group of* $(H, \cdot)$.

The core of $(\underline{H}, \cdot)$ is the set H. Moreover, scalar elements of $(H, \cdot)$ are also scalars in $(\underline{H}, \cdot)$ and any unit element of $(H, \cdot)$ is also a unit element of $(\underline{H}, \cdot)$. Finally, remark that if $(H, \cdot)$ is COW (or commutative) then $(\underline{H}, \cdot)$ is also COW (respectively, commutative).

The Attaching Elements Construction. Let $(H, \cdot)$ be a H_V-semigroup and $\{v_1, v_2, \ldots, v_n\} \cap H = \emptyset$, is an ordered set of elements outside of H, where we consider that $v_i < v_j$, when $i < j$. Then we extend the $(\cdot)$ in $\underline{H}_n = H \cup \{v_1, v_2, \ldots, v_n\}$ as follows:

$$x \cdot v_i = v_i \cdot x = v_i \ \textit{for all} \ x \in H \ \textit{and} \ i \in \{1, 2, \ldots, n\}$$

$$v_i \cdot v_j = v_j \cdot v_i = v_j \ \textit{for all} \ i < j \ \textit{and} \ v_i \cdot v_i = H \cup \{v_1, \ldots, v_{i-1}\} \ \textit{for all} \ i \in \{1, \ldots, n\}.$$

Then $(\underline{H}_n, \cdot)$ *is a h/v-group where* $(\underline{H}_n, \cdot)/\beta^* \cong \mathbf{Z}_2$ *and the last element* v_n *is single.*

4. Single elements. The core.

Theorem 9. Let $(H, \bullet)$ be an H_v-group and $x \in S_H$. For any $a \in H$, take any element $v \in H$ such that $x \in av$. Then $\beta^*(a) = \{h \in H \mid hv = x\}$.

Proof. $x \in av$, so $x = av$, which means $x = \beta^*(a) \otimes \beta^*(v)$. Thus, for all $h \in \beta^*(a)$, we have $hv = x$. Conversely, let $x = hv$. Then $x = \beta^*(h) \otimes \beta^*(v)$. Since H/β^* is a group, we have

$$\beta^*(h) = x \otimes (\beta^*(v))^{-1} = \beta^*(a), \ \ \text{so} \ \ h \in \beta^*(a).$$

Corollary. If $x \in S_H$, then $\omega_H = \{u \mid ux = x\} = \{u \mid xu = x\}$.

If $\mid \omega_H \mid = 1$, then the hypergroup is called 1-hypergroup see [1]

Theorem 10. Let $(H, \cdot)$ be an H_v-group, then $u \in \omega_H$ if and only if there exists $a \in H$ and $A \subset \beta^*(a)$ such that $uA \cap A \neq \emptyset$.

Proof. Let $u \in \omega_H$. Take $A = \beta^*(a)$ for an arbitrary $a \in H$, then

$$\varphi(u\beta^*(a)) = \varphi(u) \otimes \varphi(\beta^*(a)) = \beta^*(a).$$

Thus, $u\beta^*(a) \subset \beta^*(a)$, so $uA \cap A \neq \emptyset$.
Conversely, from $uA \cap A \neq \emptyset$ we obtain $\varphi(uA) = \varphi(A)$, so $u \in \omega_H$.

This property states that the elements of the core satisfy a generalization of the 'partial identity' property.

We recall [9] that the powers in H_v-structures are considered to be with respect to *n-ary circle hyperoperation*, that is the union of all hyperproducts is taken with all possible patterns of parentheses put on them.

Theorem 11. Let $(H, \cdot)$ be a finite H_v-group. For every element $a \in H$, there exists a power α^λ, we take the minimal one, which contains an element of a lower power, i.e., there exists α^κ such that $\alpha^\kappa \cap \alpha^\lambda \neq \emptyset$, $\kappa < \lambda$. Then $\alpha^{\lambda-\kappa} \subset \omega_H$.

Proof. From $\alpha^\kappa \cap \alpha^\lambda \neq \emptyset$, we have $\varphi(\alpha^\kappa) = \varphi(\alpha^\lambda)$ or $(\varphi(\alpha))^\kappa = (\varphi(\alpha))^\lambda$. But $(\varphi(\alpha))^\kappa$ and $(\varphi(\alpha))^\lambda$ are elements of a group, so $(\varphi(\alpha))^{\lambda-\kappa} = \omega_H$, because ω_H is the identity of H/β^*. Therefore, $\varphi(\alpha^{\lambda-\kappa}) = \omega_H$, so $\alpha^{\lambda-\kappa} \subset \omega_H$.

Remarks 12. (i) The above theorem can be used in the infinite case as well: if there are powers κ and λ with $\kappa < \lambda$ such that $\alpha^\kappa \cap \alpha^\lambda \neq \emptyset$, then we have $\varphi(\alpha^{\lambda-\kappa}) = \omega_H$, so $\alpha^{\lambda-\kappa} \subset \omega_H$.

(ii) If, moreover for some $0 < \mu < \lambda - \kappa$, the relation $\alpha^{\kappa+\mu} \cap \alpha^{\lambda} \neq \emptyset$, is valid, then we obtain $\alpha^{\lambda-\kappa-\mu} \subset \omega_H$. So, there exists a power of the element α, lower than $\lambda - \kappa$, which is contained in the core. If two successive powers α^{μ} and $\alpha^{\mu+1}$ have a common element, then $\alpha \in \omega_H$. Therefore, all the powers of the element α belong to the core.

(iii) Let A be a closed set of an H_v-group: $A \cdot A \subset A$. If $\varphi(A)$ is finite, then $A \cap \omega_H \neq \emptyset$. Indeed, from $\varphi(A)\varphi(A) \subset \varphi(A)$ it follows that $\varphi(A)$ is a closed finite subset of the quotient group H/β^*. Thus, $\varphi(A)$ is a subgroup. Since the unit element e of H/β^* is in $\varphi(A)$ and $\varphi^{-1}(e) = \omega_H$, then, $A \cap \omega_H \neq \emptyset$.

(iv) If $\alpha^{\kappa} \cap \alpha^{\lambda} \neq \emptyset$, then, for every natural number μ, we have $\alpha^{\kappa+\mu}\beta^*\alpha^{\lambda+\mu}$. Indeed, $\alpha^{\kappa} \cap \alpha^{\lambda} \neq \emptyset$ implies $\varphi(\alpha^{\kappa}) = \varphi(\alpha^{\lambda})$, so $\varphi(\alpha^{\kappa+\mu}) = \varphi(\alpha^{\lambda+\mu})$.

5. Applications.

Special classes of hypergroupoids are interesting with respect either to their hypergroup algebra or to their representations. We recall some definitions from [9].

H_v-matrix (or h/v-matrix) is called a matrix with entries of a H_v-ring (or h/v-ring). The hyperproduct of two H_v-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices:

$$\mathbf{A} \cdot \mathbf{B} = (a_{ij}) \cdot (b_{ij}) = \left\{ \mathbf{C} = (c_{ij}) \mid c_{ij} \in \oplus \sum a_{ik} \cdot b_{kj} \right\},$$

where $\oplus$ denotes the *n-ary circle hyperoperation* on the hyperaddition. The hyperproduct of H_v-matrices does not necessarily satisfy the WASS. In 2×2 H_v-matrices the 2-ary circle hyperoperation coincides with the hyperaddition in the H_v-ring.

The problem of the h/v-matrix representations is the following:

Let $(H, \cdot)$ be h/v-group. Find a h/v-ring $(R, +, \cdot)$, a set $\mathbf{M}_R = \{(a_{ij}) \mid a_{ij} \in R\}$ and a map $\mathbf{T} : H \to \mathbf{M}_R : h \to \mathbf{T}(h)$ such that $\mathbf{T}(h_1h_2) \cap \mathbf{T}(h_1)\mathbf{T}(h_2) \neq \emptyset$, $\forall\, h_1, h_2 \in \mathbf{H}$. The map $\mathbf{T}$ is called *h/v-matrix representation*. If $\mathbf{T}(h_1h_2) \subset \mathbf{T}(h_1)\mathbf{T}(h_2)$, $\forall\, h_1, h_2 \in H$, then $\mathbf{T}$ is called *inclusion*. If $\mathbf{T}(h_1h_2) = \mathbf{T}(h_1)\mathbf{T}(h_2) = \{\mathbf{T}(h) \mid h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then $\mathbf{T}$ is a *good* and in this case an induced representation $\mathbf{T}^*$ for the hypergroup algebra is obtained. If $\mathbf{T}$ is a good and one to one, then it is called *faithful*.

The cardinality of the product of two h/v-matrices is normally very big, but the problem can be simplified in several special cases as the following ones:

- The h/v-rings have 0 and 1 and, even more, if they are scalars.
- The 2×2 h/v-matrices of the lowest dimensional, non degenerate, representations.
- The strong associativity in the hyperaddition is valid, since the circle hyperoperation coincides with the hyperaddition in h/v-rings.

The main theorem of the theory of representations is [9]:
A necessary condition in order to have an inclusion representation $\mathbf{T}$ of the h/v-group $(H,\cdot)$ by $n\times n$ h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following:
For all classes $\beta^*(a)$, $a\in H$ there must exist elements $a_{ij}\in R$, $i,j\in\{1,\ldots,n\}$ such that

$$\mathbf{T}(\beta^*(a))\subset\{A=(a'_{ij})\mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,\ldots,n\}\}.$$

Analogous, to some classical ones, theorems on the characters of the h/v-representations, are proved, where the fundamental structures play crucial role.

We conclude with some constructions for low cardinality representations:

(1) The constructions appearing in section 3, especially the ones in enlarging hyperrings, can be used as entries of H_v-matrices to represent H_v-groups for which the cardinality of all hyprerproducts equals to 2^s, $s\in N$.

(2) The class of *monomial matrix representations* is based, on the ring Z_2. The enlargements of this ring, from section 3, give the following hyperrings

(i) $0\oplus 0=1\oplus 1=1\oplus v=v\oplus 1=v\oplus v=0,\ 0\oplus 1=0\oplus v=1\oplus 0=v\oplus 0=\{1,v\}$
$0\otimes 0=0\otimes 1=1\otimes 0=0\otimes v=v\otimes 0=0,\ 1\otimes 1=1\otimes v=v\otimes 1=v\otimes v=1$

(ii) $0\oplus 0=0\oplus v=1\oplus 1=v\oplus 0=\{0,v\},\ v\oplus v=0,\ 0\oplus 1=1\oplus 0=1\oplus v=v\oplus 1=1$
$1\otimes 1=1$ and in the rest cases 0.

(iii) $0\oplus 0=1\oplus 1=v\oplus v=\{0,v\},\ 0\oplus v=v\oplus 0=0,\ 0\oplus 1=1\oplus 0=1\oplus v=v\oplus 1=1$
$1\otimes 1=1$ and in the rest cases 0.

(3) ***The Attached h/v-field.*** Let $(H,\cdot)$ be a H_v-semigroup and $v\notin H$ and $(\underline{H},\cdot)$ be its attached h/v-group. Consider an element $0\notin\underline{H}$ and define in the set $\underline{H}_0=H\cup\{v,0\}$ two hyperoperations as follows:

hyperaddition $(+)$: $0+0=x+v=v+x=0,\ 0+v=v+0=x+y=v,$ $0+x=x+0=v+v=H,\ \forall\ x,y\in H.$

hypermultiplication $(\cdot)$: the hyperoperation remains the same as in $\underline{H}$ moreover

$$0\cdot 0=v\cdot x=x\cdot 0=0,\quad \forall x,y\in\underline{H}$$

Then $(\underline{H}_0,+,\cdot)$ is a h/v-field with $(\underline{H}_0,+,\cdot)/\gamma^*\cong \boldsymbol{Z}_3$. The hyperoperation $(+)$ is associative, $(\cdot)$ is WASS and weak distributive with respect to $(+)$. 0 is absorbing and single element but not scalar in $(+)$. $(\underline{H}_0,+,\cdot)$ is called the Attached h/v-field of the H_v-semigroup $(H,\cdot)$.

In attached h/v-fields $(\underline{H}_0,+,\cdot)$, in the expressions $\sum a_{ik}\cdot b_{kj}$ the terms $a_{ik}\cdot b_{kj}$ could be $0,v,x$ or H (where $x\in H$). By contrast any sum of such elements is only 0 or v or H. Thus, for finite $(\underline{H}_0,+,\cdot)$, if the entry H appears, then the cardinality is greater than one. More precisely, the cardinality is $(\text{card}H)^t$ if in t entries the set H appears.

(4) ***The 0-construction:*** Let $(H, \cdot)$ be H_v-group. Take an element $0 \notin H$ and denote $H' = H \cup \{0\}$. We define the hyperoperation $(+)$ as follows:

$$0 + 0 = 0,\ \ 0 + x = H = x + 0,\ \ x + y = 0,\ \forall\ x, y \in H,$$

and we extend the hyperoperation $(\cdot)$ in H' by setting $0 \cdot 0 = 0 \cdot x = x \cdot 0 = 0$, $\forall x \in H$. Then, $(H', +, \cdot)$ is a H_v-field with $H'/\gamma^* \cong \boldsymbol{Z}_2$ where 0 is an absorbing and single element.

This construction is useful if the cardinality of the hyperproducts of the elements, of a H_v-group needed to be represented, is equal to a power of card$H$. The H_v-groups of constant length, such as the P-hypergroups, can be also represented by these H_v-fields.

(5) The Lie-Santilli theory of isotopies was born in late 1970's, to solve problems mainly in Hadronic Mechanics. The founder of this theory, R.M. Santilli, [3], proposed a 'lifting' of the n-dimensional trivial unit matrix of a conventional theory into a nowhere singular, symmetric, real-valued, positive-defined and n-dimensional new matrix. The original theory is reconstructed to admit the new matrix as left and right unit. The 'isofields' needed correspond into the hyperstructure theory to the so-called *e-hyperfields*. The H_v-fields or h/v-fields can give a large number of e-hyperfields. Thus: Let $(\underline{H}_0, +, \cdot)$ be the attached h/v-field of the H_v-semigroup $(H, \cdot)$. If the H_v-semigroup $(H, \cdot)$ has a left and right scalar unit e then $(\underline{H}_0, +, \cdot)$ is a e-hyperfield. The element $e \notin H$ is the unit which can be used in any 'iso-lifting'.

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