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Arithmetical properties of the exponents of Catalan's equation

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Abstract

We consider Catalan's equation $x^m - y^n = \pm 1$ (with $2 \le m < n$ and $x,\, y > 1$). We prove that m is prime and that n has at most two non-trivial divisors.

In 1843, E. Catalan posed the following famous problem: are there any consecutive positive integers other than 8 and 9 which are both powers? This is equivalent to consider "Catalan's equation"

$$x^m - y^n = \pm 1, \qquad \text{with } 2 \le m < n \text{ and } x,\, y > 1, \tag{1}$$

which is the main thema of the book of P. Ribenboim [4] and where the interested reader can find a lot of information. The solution $(x, y, m, n) = (3, 2, 2, 3)$ is called the trivial one.

Our study is based on the two following results.

Proposition 1 *[2] If r is a prime number and s an integer ≥ 2 such that the equation*

$$x^r - y^s = \pm 1 \tag{2}$$

has a non-trivial solution in positive integers x and y then

$$s < 2.77\, r\, (\log(s/\log r) + 2.333)^2 \log r. \tag{3}$$

Remarks.

1) In [2], this inequality is stated only in the case that s is also prime, but the proof does not use this assumption.

2) Proposition 1 shows that, given r, the exponent s is bounded in terms of s, we denote this upper bound by $s \le Q(r)$. One can check that $Q(r) < r^2$ for $r > 10^5$.

Proposition 2 *[3] If r and s are integers ≥ 2 such that equation (2) has a non-trivial solution in positive integers x and y then* $\min\{r, s\} > 10^5$.

These two propositions imply easily the following result, which was published in [1]:

Theorem 1 *If Catalan's equation*

$$x^m - y^n = \pm 1 \quad \textit{with } 2 \leq m < n, \textit{ and } x, y > 1$$

has a solution then m is prime.

Proof.— Consider equation (1) and suppose that m is not prime, then m has a prime divisor p with $p^2 \leq m$. Relation (1) implies $(x^{m/p})^p - y^n = \pm 1$, thus (3) holds with the pair $(r, s) = (p, n)$. By Remark 2, we have $n < p^2$, and we get the contradiction $p^2 \leq m < n < p^2$.

One can also get information on the exponent n:

Theorem 2 *If Catalan's equation*

$$x^m - y^n = \pm 1 \quad \textit{with } 2 \leq m < n, \textit{ and } x, y > 1$$

has a solution then n admits at most two non-trivial divisors.

Proof.— By Theorem 1 we know that m is a prime. Let q be the smallest prime factor of n. Suppose that $\Omega(n) \geq 3$, then, applying Proposition 1 twice, we get

$$q \leq n^{1/3} \leq Q(m)^{1/3} \leq Q(Q(q))^{1/3}.$$

An easy computational verification shows that this leads to a contradiction for $q > 10^5$.

References

[1] M. MIGNOTTE, Une remarque sur l'équation de Catalan, Proceedings of the Zakopane Conference, Poland, Jul. 97, ed. K. Győry.

[2] M. MIGNOTTE et Y. ROY, Catalan's equation has no new solution with either exponent less than 10651, Experimental Mathematics, 4, 1995, p. 259–268.

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[4] P. RIBENBOIM, *Catalan's*

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